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FUZZY PRE-CONNECTEDNESS AND FUZZY α-CONNECTEDNESS IN FUZZY BICLOSURE SPACES

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Abstract: The aim of this paper is to extend the concepts of fuzzy pre-connectedness and fuzzy α- connectedness from fuzzy closure spaces to fuzzy biclosure spaces. We explore the basic fundamental properties of fuzzy pre-connectedness and fuzzy α- connectedness in fuzzy biclosure spaces. Here we consider fuzzy closure operator as a generalization of Birkhoff closure operator.

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Keywords: fuzzy pre-separated sets, fuzzy α- seperated sets, fuzzy pre-connectedness, fuzzy α-connectedness.

1. Introduction

Closure operators were introduced by Čech and Birkhoff independently. In 1985 Mashhour and Ghanim [13] generalized the Čech closure operator in fuzzy setting. After that in 1994 Srivastava et.al. [19] generalized the Birkhoff closure operators in fuzzy setting. The concept of fuzzy closure space was first introduced by A.S. Mashhour and Ghanim [12]. A fuzzy set X together with two fuzzy closure operators is known as fuzzy biclosure spaces. Here we introduce separation axioms, fuzzy pre-connectedness and fuzzy α- connectedness in fuzzy biclosure spaces using fuzzy pre-open sets and fuzzy α- open sets in different manner and obtain some results.

2. Preliminaries

The notion of fuzzy set was introduced by L. A. Zadeh (1965) in his classical paper [21]. A fuzzy set 'A' in a non-empty set X is a mapping from X to [0,1]. A fuzzy point x_r is a fuzzy set in X taking value $r \in (0,1)$ at x and zero elsewhere. A fuzzy point x_r is said to belong to a fuzzy set A i.e. $x_r \in A$ iff $r \leq A(x)$ [15]. A fuzzy singleton x_r is a fuzzy set in X taking value $r \in (0,1]$ at x and 0 elsewhere. A non-empty set X together with two fuzzy topologies τ_1, τ_2 is called fuzzy bitopological space and it is denoted by $(X, \tau_{1,} \tau_2)$.

A fuzzy point x_r is said to be quasi-coincident with a fuzzy set A denoted by $x_r qA$ iff $r + A(x) > 1$. A fuzzy set A is said to be quasi-coincidence with another fuzzy set B denoted by $AqB\ iff\ \exists\ x \in X$ such that $A(x) + B(x) > 1$. Similarly we say that $A\bar{q}B$ iff $A \subseteq coB\ i.e.\ A(x) + B(x) \leq 1$. Obviously if $A\ and\ B$ are quasi-coincident at x both $A(x)\ and\ B(x)$ are not zero and hence $A\ and\ B$ intersect at x. Here we follow the Lowen's definition [10] of fuzzy topology. According to him a family τ of fuzzy sets of a non-empty X is said to form a fuzzy topology on X if it is closed under arbitrary union, finite intersection and contains all constant fuzzy sets. The members of τ are called fuzzy open sets and their complements are called fuzzy closed sets.

The concept of closure operator was given by Čech [6] and Birkhoff [7] independently. Researchers have done many works using Čech closure operator. We have done our research using the Birkhoff closure operator. We mention the definition of fuzzy biclosure operator as a generalization of closure operator mentioned in [20]. We mention here all these definitions:

Definition 2.1([6]). Let X be a non-empty set. A Čech closure operator on X is a map $C: P(X) \to P(X)$ which satisfies:

1. $C(\emptyset) = \emptyset$,
2. $A \subseteq C(A),\ \forall A \subseteq P(X)$,
3. $C(A \cup B) = C(A) \cup C(B), \forall A, B \subseteq P(X)$.

Definition 2.2 ([3]). A closure operator on a non-empty set X is a function $C : P(X) \to P(X)$ is said to be Birkhoff closure operator on X if it satisfies the following conditions:

1. $C(\emptyset) = \emptyset$,
2. $A \subseteq C(A),\ \forall A \subseteq P(X)$,
3. $A \subseteq B \Rightarrow C(A) \subseteq C(B), \forall A, B \subseteq P(X)$,
4. $C(C(A)) = C(A)$.

Definition 2.3 ([12]). A function $c : I^X \to I^X$ is called a fuzzy closure operation on a non-empty set X if it satisfies the following conditions:

1. $c(\emptyset) = \emptyset$,
2. $A \subseteq c(A), \forall A \in I^X$,
3. $c(A \cup B) = c(A) \cup c(B), \forall A, B \in I^X$.

Definition 2.4 ([19]). A function $c : I^X \to I^X$ is called fuzzy closure operation on a non-empty set X if it satisfies the following conditions:

1. $c(\underline{\alpha}) = \underline{\alpha}; \alpha \in [0, 1]$
2. $A \subseteq c(A)$,
3. $A \subseteq B \Rightarrow c(A) \subseteq c(B), \forall A, B \in I^X$
4. $c(c(A)) = c(A)$.

Definition 2.5. A function $c_i : I^X \longrightarrow I^X (i = 1,2)$ is called a fuzzy biclosure operator on a non-empty set X if the following postulates are satisfies:

1. $c_i(\underline{\alpha}) = \underline{\alpha}$; $\alpha \in [0,1]$,
2. $A \subseteq c_i(A)$,
3. $A \subseteq B \Rightarrow c_i(A) \subseteq c_i(B)$,
4. $c_i(c_i(A)) = c_i(A)$.

The structure (X, c_1, c_2) is called a fuzzy biclosure space.

We mention here some definitions in fuzzy closure spaces which will be generalized to fuzzy biclosure spaces later:

A fuzzy closure space (Y, c^*) is said to be a subspace of (X, c) if $Y \subseteq X$ and $c^*(A) = c(A) \cap Y$ for each subset $A \subseteq Y$. Let (X, c) and (Y, c^*) be two fuzzy closure spaces then if map $f: (X, c) \to (Y, c^*)$ is continuous, then $f^{-1}(F)$ is closed subset of (X, c) for every closed subset F of (Y, c^*). A map $f: (X, c) \to (Y, c^*)$ is said to be fuzzy continuous if $f(cA) \subseteq c^* f(A)$ for every subset $A \subseteq X$.

A map $f: (X, c) \to (Y, c^*)$ is said to be fuzzy closed (resp. fuzzy open) if $f(F)$ is fuzzy closed (resp. fuzzy open) subset of (Y, c^*) whenever F is a fuzzy closed (resp. fuzzy open) subset of (X, c).

The product of a family $\{(X_\alpha, c_\alpha) : \alpha \in I\}$ of fuzzy closure spaces, denoted by $\prod_{\alpha \in I}(X_\alpha, c_\alpha)$, is the fuzzy closure space $(\prod_{\alpha \in I} X_\alpha, c)$ where $\prod_{\alpha \in I} X_\alpha$ denotes the cartesian product of sets $X_\alpha, \alpha \in I$, and c is the fuzzy closure operator generated by the projections $\pi_\alpha: \prod_{\alpha\in I} X_\alpha \to X_\alpha, \alpha \in I$ i.e. is defined by $cA = \prod_{\alpha\in I} c_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha\in I} X_\alpha$.

We generalized these concepts in fuzzy biclosure space as:

Definition 2.6. Let (X, c_1, c_2) be a fuzzy biclosure space. If the closure operator ful fill the condition $c_i(A \cup B) = c_i(A) \cup c_i(B)$ then it is said to satisfy the additive property.

Definition 2.7. The fuzzy biclosure space (X, c_1, c_2) is said to be coarser than (X, c_1^*, c_2^*) if $c_i(A) \subseteq c_i^*(A)$ for all $i = 1,2$ *and* $A \in I^X$.

Definition 2.8 ([2]). A subset A of a fuzzy biclosure space (X, c_1, c_2) is called fuzzy closed if $c_1c_2A = A$. The complement of fuzzy closed set is called fuzzy open.
Clearly, A is a fuzzy closed subset of a fuzzy biclosure space (X, c_1, c_2) if and only if A is a fuzzy closed subset of both (X, c_1) and (X, c_2).
Let A be a fuzzy closed subset of a fuzzy biclosure space (X, c_1, c_2). The following conditions are equivalent

1. $c_2c_1A = A$
2. $c_1A = A, c_2A = A$.

A fuzzy biclosure space (Y, c_1^*, c_2^*) is said to be a subspace of (X, c_1, c_2) if $Y \subseteq X$ and $c_i^*(A) = c_i(A) \cap Y$ for each subset $A \subseteq Y$. Let (X, c_1, c_2) and (Y, c_1^*, c_2^*) be two fuzzy biclosure spaces then if map $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ is pairwise continuous, then $f^{-1}(F)$ is closed subset of (X, c_1, c_2) for every closed subset F of (Y, c_1^*, c_2^*). A map $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ is said to be fuzzy pairwise continuous if $f(c_1A) \subseteq c_2f(A)$ for every subset $A \subseteq X$.

A map $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ is said to be fuzzy open (resp. fuzzy closed) if $f(F)$ is fuzzy open (resp. fuzzy closed) subset of (Y, c_1^*, c_2^*) whenever F is a fuzzy open (resp. fuzzy closed) subset of (X, c_1, c_2).

The product of a family $\{(X_\alpha, c_{1_\alpha}, c_{2_\alpha}) : \alpha \in I\}$ of fuzzy biclosure spaces, denoted by $\prod_{\alpha\in I}(X_\alpha, c_{1_\alpha}, c_{2_\alpha})$, is the fuzzy biclosure space $(\prod_{\alpha\in I} X_\alpha, c_1, c_2)$ where $\prod_{\alpha\in I} X_\alpha$ denotes the cartesian product of sets $X_\alpha, \alpha \in I$, and c_1, c_2 are the fuzzy closure operators generated by the projections $\pi_\alpha: \prod_{\alpha\in I} X_\alpha \to X_\alpha, \alpha \in I$ i.e. is defined by $c_iA = \prod_{\alpha\in I} c_{i_\alpha} \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha\in I} X_\alpha$.

3. Fuzzy pre-open sets in fuzzy biclosure spaces

The concept of pre-open set was introduced by A.S. Mashhour [13] as "A set A in a topological space X is called pre-open iff $A \subseteq int(cl(A))$. The notion of pre-open set in closure space was introduced by Rao, Gowri and Swaminathan [7] and it was further defined in biclosure spaces by Rao and Gowri)[8].

Definition 3.1 ([1]). A subset A of a fcs (X, c) is said to be fuzzy pre-open if $A \subseteq int(cl(A))$. The complement of fuzzy pre-open sets is called fuzzy pre-closed if $A \subseteq cl(int(A))$.
Definition 3.2 ([18]). Let (X, c_1, c_2) be a fuzzy biclosure space and the fuzzy pre closure of a fuzzy set A in X is defined as;
$fuzzy\ pre\text{-}c_i(A) = \cap \{B: B\ is\ fuzzy\ pre\ closed\ set\ and\ B \supset A\}$
Similarly the fuzzy pre interior of a fuzzy set A in X is defined as:
$fuzzy\ pre\text{-}int_i(A) = \cup \{B: B\ is\ fuzzy\ pre\ open\ set\ and\ B \subset A\}$
Arbitrary union of fuzzy pre-open sets is fuzzy pre-open.

3.1. Fuzzy pairwise pre-separation axioms in fuzzy biclosure spaces

In this section we define the concept of fuzzy pre- separation axioms using fuzzy pre-open sets in fuzzy biclosure spaces:

Definition 3.1.1. A fuzzy biclosure space (X, c_1, c_2) is said to be

1. Fuzzy pairwise pre T_0 $if\ \forall\, x, y \in X, x \neq y\ \exists$ a fuzzy pre-open set $U\ such\ that\ U(x) \neq U(y)$

2. Fuzzy pairwise pre T_1 $if\ \forall\, x, y \in X, x \neq y\ \exists$ fuzzy pre-open sets $U, V\ in\ X\ such\ that\ U(x) = 1, U(y) = 0\ and\ V(x) = 0, V(y) = 1,$

3. Fuzzy pairwise weakly pre T_1 if $\exists$ a c_1-fuzzy pre-open set or a c_2-fuzzy pre-open set $U\ such\ that\ U(x) = 1, U(y) = 0,$

4. Fuzzy pairwise pre T_2 if for every pair of distinct fuzzy points $x_r, y_s\ in\ X$ their exists fuzzy pre-open sets $U\ and\ V\ such\ that\ x_r \in U, y_s \in V\ and\ U \cap V = \emptyset,$

5. Fuzzy pairwise pre-regular if for each fuzzy point x_r and each fuzzy closed set $F\ such\ that\ x_r \bar{q} F$ $\exists$ fuzzy pre-open sets $U\ and\ V\ such\ that\ x_r \subseteq U, F \subseteq V\ and\ U \bar{q} V$,

6. Fuzzy pairwise pre-normal if for every pair of fuzzy closed set $F_1\ and\ F_2\ such\ that\ F_1 \bar{q} F_2,$ $\exists$ fuzzy pre-open sets $U, V\ such\ that\ F_1 \subseteq U\ and\ F_2 \subseteq V\ and\ U \bar{q} V.$

Clearly fuzzy pairwise pre $T_2 \Rightarrow$ fuzzy pairwise pre $T_1 \Rightarrow$ fuzzy pre pairwise T_0 but not conversely.

3.2. Fuzzy pairwise pre-continuous maps in fuzzy biclosure spaces

Fuzzy pairwise pre-continuous maps was introduced by R. Srivastava [20]. We introduce and study two more definitions using fuzzy pre-open sets and compare all the definitions with each other.

Let (X, c_1, c_2) and (Y, c_1, c_2) be two fuzzy biclosure spaces then:

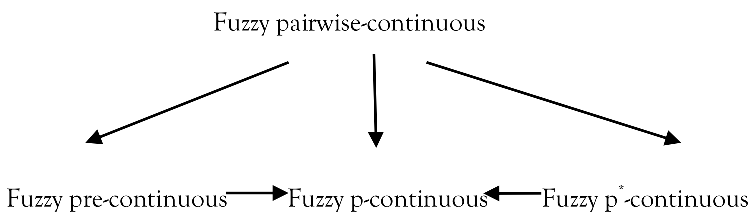
Definition 3.2.1 ([1]). A map $f : X \longrightarrow Y$ is said to be fuzzy pairwise continuous if $f^{-1}(V)$ is fuzzy open in X whenever V is fuzzy open set in Y.

Definition 3.2.2 ([1]). A map $f : X \longrightarrow Y$ is said to be fuzzy pairwise p-continuous if $f^{-1}(V)$ is pre-open in X whenever V is pre-open set in Y.

Definition 3.2.3. A map $f : X \longrightarrow Y$ is said to be fuzzy pairwise pre-continuous if $f^{-1}(V)$ is open in X whenever V is pre-open set Y.

Definition 3.2.4. A map $f : X \longrightarrow Y$ is said to be fuzzy pairwise p*-continuous if $f^{-1}(V)$ is pre-open in X whenever V is open set in Y.

Comparing all the above definitions:



We got some more pre-open maps given below:

Definition 3.2.5. A mapping $f: X \to Y$ is called fuzzy open if $f(v)$ is fuzzy open set in Y whenever v is fuzzy open in X.

Definition 3.2.6. A mapping $f: X \to Y$ is called fuzzy p-open map if $f(v)$ is fuzzy pre-open in Y whenever v is fuzzy pre-open in X.

Definition 3.2.7. A mapping $f: X \to Y$ is called fuzzy p^*-open map if $f(v)$ is fuzzy pre-open in Y whenever the fuzzy set v is open in X.

Definition 3.2.8. A mapping $f: X \to Y$ is called fuzzy pre-open map if $f(v)$ is fuzzy open in Y whenever v is fuzzy pre-open in X.

3.3 Fuzzy pre-connectedness in fuzzy biclosure spaces

In this section we introduce the concept of q-separated sets. This concept is earlier introduced by Ming-Ming [15] in 1980.

Definition 3.3.1 ([15]). Two fuzzy sets $A_1 \, and \, A_2$ in fuzzy topological space (X, T) are said to be separated iff $\exists\, U_i \in T (i = 1,2) : U_i \supset A_i \, and \, U_1 \cap A_2 = \emptyset = U_2 \cap A_1$.

Definition 3.3.2 ([15]). The fuzzy sets $A_1 \, and \, A_2$ in fuzzy topological space (X, T) are said to be Q-separated iff $\exists\, T$ closed sets $H_i \; i = 1,2$ such that $H_i \supset A_i$ and $H_1 \cap A_2 = \emptyset = H_2 \cap A_1$.

It is obvious that $A_1 \, and \, A_2$ are Q-Separated iff $\bar{A}_1 \cap A_2 = \emptyset = \bar{A}_2 \cap A_1$.

In a similar way we define c_i-q-$separated$ sets in fuzzy biclosure space.

Definition 3.3.3. Two fuzzy sets $A \, and \, B$ in a fcs (X, c) are said to be q-separated iff $cl(A)\bar{q}B$ and $A\bar{q}cl(B)$.

Definition 3.3.4. A fuzzy set Y in (X, c) is called disconnected iff there exist two non-empty sets $A \, and \, B$ in the subspace Y (i.e supp Y_0) such that $A \, and \, B$ are q-separated and $Y = A \cup B$. A fuzzy set is said to be connected iff it is not disconnected.

Definition 3.3.5. Two fuzzy sets $A \, and \, B$ in a fbcs (X, c_1, c_2) are said to be c_i-q-separated (or simply q-separated) iff $c_i(A)\bar{q}B \, and \, c_i(B)\bar{q}A$.

Definition 3.3.6. Two non-empty fuzzy pre-open sets $A \, and \, B$ in a fbcs (X, c_1, c_2) are said to be fuzzy pre-c_i-q-separated iff p-$c_i(A)\bar{q}B$ and p-$c_i(B)\bar{q}A$.

We introduce here the definition of pre-connectedness in fuzzy biclosure space:

Definition 3.3.7. A fuzzy biclosure space (X, c_1, c_2) is said to be fuzzy pre-disconnected iff there exist two non-empty fuzzy pre-open sets $A \, and \, B$ such that $X = A \cup B$ $(Here \, suppX_0)$ such that p-$c_i(A)\bar{q}B$ and $A\bar{q}p$-$c_i(B)$.

Theorem 3.3.1. *The pairwise p-continuous onto image of a fuzzy pre-connected bi-closure space having additive property is fuzzy pre-connected biclosure space with additive property.*

Proof. Let $X \, and \, Y$ be two fuzzy biclosure spaces having additive property and let X be fuzzy pre-connected. Let us suppose that Y is not fuzzy pre-connected then Y is fuzzy pre disconnected. Then $\exists$ two non-empty fuzzy pre-open sets

$A\ and\ B$ in the subspace $Y(suppY_0)$ such that $A\ and\ B$ are pre q-separated and $Y = A \cup B\ (suppY_0)$ or $p\text{-}c_i(A)\bar{q}B\ and\ A\bar{q}p\text{-}c_i(B)$ since $f\ is$ fuzzy pairwise p-continuous $f^{-1}(A)\ and\ f^{-1}(B)$ are subsets of X therefore $X = f^{-1}(A) \cup f^{-1}(B)$ where $p\text{-}c_i(f^{-1}(A))\bar{q}f^{-1}(B)$ and $f^{-1}(A)\bar{q}p\text{-}c_i(f^{-1}(B))$. We know from the additive property that $c_i(A \cup B) = c_i(A) \cup c_i(B)$ then $f^{-1}Y = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ which is a contradiction since $f^{-1}(A)\ and\ f^{-1}(B)$ are pre q-separated. It means our assumption is wrong. Thus Y is fuzzy pre-connected biclosure space having additive property.

Theorem 3.3.2. *Let (X, c_1, c_2) be fuzzy pre-disconnected let $c_i \subset c_i^*(i = 1,2)$ then (X, c_1^*, c_2^*) is fuzzy pre-disconnected.*

Proof. Let (X, c_1, c_2) be fuzzy pre-disconnected fbcs then $\exists$ two non-empty fuzzy pre-open sets $A\ and\ B$ in the subspace $Y(suppY_0)$ such that $Y = A \cup B$ where $A\ and\ B$ are $prec_i$-separated. Since $c_i \subset c_i^*$ then $A\ and\ B$ are $prec_i^*$-separated also. Thus we have two non-empty sets $A\ and\ B$ in the space $Y(suppY_0)$ such that $Y = A \cup B$ where $A\ and\ B$ are $prec_i^*$-separated also. Hence (X, c_1^*, c_2^*) is pre-disconnected.

Theorem 3.3.3. *Let (X, c_1, c_2) be a fuzzy pre-connected. Let $c_i^* \subset c_i$ then (X, c_1^*, c_2^*) is also fuzzy pre-connected.*

Proof. Let (X, c_1, c_2) be a fuzzy pre-connected fuzzy biclosure space. Then it cannot be written as union of two non-empty pre q-separated sets. Let $c_i^* \subset c_i$ and suppose that (X, c_1^*, c_2^*) is fuzzy pre-disconnected. Since (X, c_1^*, c_2^*) is fuzzy pre-disconnected $\exists$ two non-empty pre-open sets $A\ and\ B$ in the subspace $Y(suppY_0)$ such that $Y = A \cup B$ where A and B are $pre\ c_i$- separated also. Then (X, c_1, c_2) is also fuzzy pre-disconnected. Since $\exists$ two non-empty pre-open sets $A\ and\ B$ such that $Y = A \cup B\ (suppY_0)$ where $A\ and\ B$ are $pre\ c_i$-separated sets which is a contradiction. Hence (X, c_1^*, c_2^*) is pre-connected.

4. Fuzzy α-open sets in fuzzy biclosure spaces

The α-open sets in topological space was introduced by Njåstad [16] and fuzzy α-open sets introduced by Singhal, Rajvanshi [18] and Bin Shahana [4].

Definition 4.1 ([4]). A subset A of a fuzzy closure space (X, c) is said to be fuzzy α-open if $A \subseteq int\ cl(\ int(A))$. The complement of fuzzy α- open sets is called fuzzy α- closed set so, a fuzzy set A is said to be fuzzy α- closed set if $cl\ int(cl(A)) \subseteq A$.

Definition4.2. Let (X, c_1, c_2) be a fuzzy biclosure space and the fuzzy α-closure of a fuzzy set A in X is defined as:

$fuzzy\ \alpha\text{-}c_i(A) = \cap \{B: B\ is\ fuzzy\ \alpha\text{-}closed\ set\ and\ B \supset A\}$

Similarly the fuzzy α-interior of a fuzzy set A in X is defined as:

$fuzzy\ \alpha\text{-}int_i(A) = \cup \{B: B\ is\ fuzzy\ \alpha\text{-}open\ set\ and\ B \subset A\}$

Definition4.3. For a fuzzy set u in X we define the interior of u as; $i_\alpha(u) = \cup\{v : u \subseteq v,\ \ v\ is\ a\ fuzzy\ \alpha\text{-}\ open\ set\}$.

Let X be a fuzzy biclosure space, $u \subseteq v \subseteq X$ then we have the following results :

1. $i_\alpha u$ is a fuzzy $\alpha\text{-}open$ set in X,
2. $i_\alpha u \subseteq u$,
3. $i_\alpha u \subseteq i_\alpha v$,
4. $i_\alpha u = i_\alpha(i_\alpha u)$,
5. A is a fuzzy $\alpha\text{-}open$ set iff $u = i_\alpha u$.

Definition 4.4. A fuzzy set A in a fbcs X is said to be a fuzzy α-$Neighbourhood$ of a point x of X iff there exists a fuzzy α-$open$set B in X such that $B \subseteq A$ and $A(x) = B(x) > 0$.

Theorem 4.1. *In a fuzzy biclosure space* (X, c_1, c_2) *we have*

1. An arbitrary union of fuzzy α-open *sets is a fuzzy* α*-open set.*

2. A finite intersection of fuzzy α-open *sets is fuzzy* α*-open set.*

3. An arbitrary intersection of fuzzy α*-closed sets is* α*-closed set.*

4. A finite union of fuzzy α*-closed sets is* α*-closed set.*

Proof. Their proofs are straight forward so omitted.

Theorem 4.2. *If* A *and* B *are fuzzy* α-$Neighbourhood$ *of* x *then* $A \cap B$ *is also a fuzzy* α-$Neighbourhood$ *of* x.

Proof. Since A and B are fuzzy α-$Neighbourhood$ of x there exist fuzzy α-$open$ sets C and$D$ such that $C \subseteq B, D \subseteq A, B(x) = C(x) > 0$ and $A(x) = D(x) > 0$. Then $A \cap D$ is a fuzzy α-$open$set in X.

4.1. Fuzzy α-separation axioms

In this section we define the concept of fuzzy α-separation axioms in fuzzy biclosure spaces:

Definition 4.1.1. A fuzzy biclosure space (X, c_1, c_2) is said to be

1. Fuzzy pairwise α T_0 if $\forall\, x, y \in X, x \neq y$ $\exists$ a fuzzy α-open set $U\ such\ that\ U(x) \neq U(y)$

2. Fuzzy pairwise α T_1 if $\forall\, x, y \in X, x \neq y$ $\exists$ fuzzy α-open sets $U, V\ in\ X\ such\ that\ U(x) = 1, U(y) = 0\ and\ V(x) = 0, V(y) = 1,$

3. Fuzzy pairwise Weakly α T_1 if $\exists$ a c_1-fuzzy α-open set or a c_2-fuzzy α-open set $U\ such\ that\ U(x) = 1, U(y) = 0,$

4. Fuzzy pairwise α T_2 if for every pair of distinct fuzzy points $x_r, y_s\ in\ X$ their exists fuzzy α-open sets $U\ and\ V\ such\ that\ x_r \in U, y_s \in V\ and\ U \cap V = \emptyset,$

5. Fuzzy pairwise α-regular if for each fuzzy point x_r and each fuzzy closed set $F\ such\ that\ x_r \bar{q} F$ $\exists$ fuzzy α-open sets $U\ and\ V\ such\ that\ x_r \subseteq U, F \subseteq V\ and\ U \bar{q} V$,

6. Fuzzy pairwise α-normal if for every pair of fuzzy closed set $F_1\ and\ F_2\ such\ that\ F_1 \bar{q} F_2,$ $\exists$ fuzzy α-open sets $U, V\ such\ that\ F_1 \subseteq U\ and\ F_2 \subseteq V\ and\ U \bar{q} V.$

Clearly fuzzy pairwise α $T_2 \Rightarrow$ fuzzy pairwise α $T_1 \Rightarrow$ fuzzy pairwise α T_0 but not conversely.

4.2. Fuzzy pairwise α- continuous maps and α- open maps in fuzzy bi-closure spaces

The concepts of fuzzy biclosed maps, pairwise fuzzy bicontinuous maps and generalized fuzzy continuous maps in fbcs were introduced by Navalakhe [12],[13]. Pairwise continuity between fuzzy closure spaces was introduced by Azad [1]. Further using fuzzy semi-open sets Azad [1] introduced and studied fuzzy pairwise s-continuous mapping between fuzzy closure spaces. We introduce and study two more definitions using fuzzy α-open sets and compare all the definitions with each other. Let (X, c_1, c_2) and (Y, c_1, c_2) be two fuzzy biclosure spaces then:

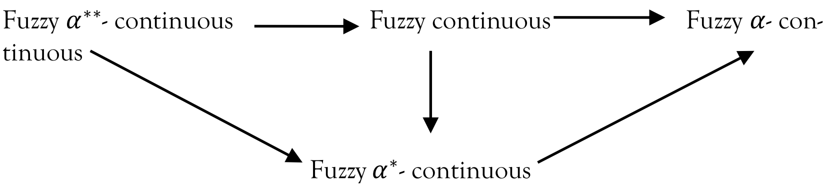
Definition 4.2.1 ([7]). A map $f : X \longrightarrow Y$ is said to be fuzzy pairwise continuous if $f^{-1}(V)$ is fuzzy open in X whenever V is fuzzy open set in Y.

Definition 4.2.2. A map $f : X \longrightarrow Y$ is said to be fuzzy pairwise α-continuous if $f^{-1}(V)$ is open in X whenever V is α-open set Y.

Definition 4.2.3. A map $f : X \longrightarrow Y$ is said to be fuzzy pairwise α^*- continuous if $f^{-1}(V)$ is α-open in X whenever V is α-open set in Y.

Definition 4.2.4. A map $f : X \longrightarrow Y$ is said to be fuzzy pairwise α^{**}- continuous if $f^{-1}(V)$ is α-open in X whenever V is open set in Y.

Comparing all the above definitions as:



Now we define some more definitions of continuity which are weaker than previous one:

Definition 4.2.5. A map $f : X \longrightarrow Y$ is said to be fuzzy α-open if $f(v)$ is fuzzy α- open in Y whenever the fuzzy set v is open in X.

Definition 4.2.6. A map $f : X \longrightarrow Y$ is said to be fuzzy α^*-open if $f(v)$ is fuzzy α- open in Y whenever v is fuzzy α- open in X.

Definition 4.2.7. A map $f : X \longrightarrow Y$ is said to be fuzzy α^{**}-open if $f(v)$ is fuzzy open in Y whenever v is fuzzy α- open in X.

Theorem 4.2.1. *Let X_1, X_2, Y_1 and Y_2 be fuzzy biclosure spaces. Let $f: X_1 \to Y_1$, $f: X_2 \to Y_2$ be two mappings. If f_1, f_2 are fuzzy α^{**}-continuous mappings then $f_1 \times f_2$ is also fuzzy α^{**}-continuous mapping here X_1 is product related to X_2.*
Proof. Let G be a fuzzy open set in $Y_1 \times Y_2$ then there exists fuzzy open sets G_i and H_i in $Y_1, and\ Y_2$ respectively such that $G = \cup\, (G_i \times H_i)$. From Lemma 2.1 and 2.3 of Azad [1], we have $(f_1 \times f_2)^{-1}(G) = sup_{i,j}[(f_1 \times f_2)^{-1}(G_i \times H_i)] = sup_{i,j}[{f_1}^{-1}(G_i) \times {f_2}^{-1}(H_i)]$ since f_1, f_2 are fuzzy α^{**}- continuous ${f_1}^{-1}(G_i)$ and ${f_2}^{-1}(H_i)$ are fuzzy α- open sets using theorem 1.5 and 1.4(a) we have $(f_1 \times f_2)(G)$ is a fuzzy α- $open$ set. Hence $f_1 \times f_2$ is fuzzy α^{**}- continuous.

Theorem 4.2.2. *Let X and Y be fuzzy biclosure spaces. Let $f: X \to Y$ be a mapping and $g: X \to X \times Y$ be a graph of f if g is fuzzy α^{**}- continuous then f is also.*

Proof. Let G be a fuzzy open set in Y then using lemma 2.4 due to Azad [1]. We have $f^{-1}(G) = X \cap f^{-1}(G) = g^{-1}(X \times G)$, since g is fuzzy α^{**}- continuous and $X \times G$ is a fuzzy open set in $X \times Y$. $f^{-1}(G)$ is a fuzzy α-open set in X. Hence f is fuzzy α^{**}- continuous.

4.3. Fuzzy α- connectedness in fuzzy biclosure spaces

Definition 4.3.1. A fuzzy set in a fuzzy closure space is said to be fuzzy α- disconnected iff there does not exist two non-empty α- open sets A and B in the subspace Y_0 (supp Y) such that $Y = A \cup B$ and A, B are q- separated.

Theorem 4.3.1. *The fuzzy α^*- continuous onto image of a fuzzy α- $connected$ biclosure space which has additive property also is fuzzy α- $connected$ biclosure space with additive property.*

Proof. Let us assume X and Y be two fuzzy biclosure spaces with additive property. Also let X be fuzzy α- connected and Y is not fuzzy α- connected then Y is fuzzy α- disconnected. So there exists two non-empty fuzzy α- open sets A and B in the subspace $Y(suppY_0)$ such that A and B are α- q-separated and $Y = A \cup B(suppY_0)$ since f is fuzzy α^*- continuous onto function so $f^{-1}(A)\ and\ f^{-1}(B)$ are subsets of X therefore $X = f^{-1}(A) \cup f^{-1}(B)$ where α-$c_i(f^{-1}(A))\bar{q}f^{-1}(B)$ and $f^{-1}(A)\bar{q}\ \alpha$-$c_i(f^{-1}(B))$. By using the additive property we can write $c_i(A \cup B) = c_i(A) \cup c_i(B)$ then $f^{-1}Y = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ which is a contradiction since $f^{-1}(A)\ and\ f^{-1}(B)$ are α-q-separated. It means our assumption is wrong. Thus Y is fuzzy α-connected biclosure space having additive property

Theorem 4.3.2. *Let (X, c_1, c_2) be fuzzy α- $disconnected$ let $c_i \subset c_i^*(i = 1,2)$ then (X, c_1^*, c_2^*) is α- $disconnected.$*

Proof. Let (X, c_1, c_2) be fuzzy α- disconnected then $\exists$ two non-empty fuzzy α-open sets $A\ and\ B$ in the subspace $Y(suppY_0)$ such that $Y = A \cup B$ where $A\ and\ B$ are α-c_i-separated. Since $c_i \subset c_i^*$ then $A\ and\ B$ are α- c_i^*-separated also. Thus we have two non-empty sets $A\ and\ B$ in the space $Y(suppY_0)$ such that $Y = A \cup B$ where $A\ and\ B$ are α- c_i^*- separated also. Hence (X, c_1^*, c_2^*) is α- disconnected.

Theorem 4.3.3. *Let (X, c_1, c_2) be a fuzzy α- connected. Let $c_i^* \subset c_i(i = 1,2)$ then (X, c_1^*, c_2^*) is also fuzzy α- $connected.$*

Proof. Let us suppose (X, c_1^*, c_2^*) is fuzzy α- disconnected then there exists two non-empty α- open sets A and B in the subspace $Y(suppY_0)$ such that $Y = A \cup B$ where A and B are α-c_i-separated also then (X, c_1, c_2) is also fuzzy α-disconnected since $c_i^* \subset c_i$ which is a contradiction. Hence (X, c_1^*, c_2^*) *is also fuzzy α- $connected.$*

Conclusion

Here we have introduced and studied separation axioms and connectedness using fuzzy pre-open sets and fuzzy α-open sets in fuzzy biclosure space. We defined fuzzy pre separated sets and fuzzy α- separated sets in a fuzzy biclosure space. Though these are weaker than open sets but they play very vital role in various fields. Their applications can be seen in fuzzy topology, nano topology and medical field.

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