

On Variational Iteration Method For Solving
Partial Differential Equations And Its Applications

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Abstract

In this paper, He's variational iteration method (VIM) has been used to solve partial differential equations. This method is based on Lagrange multiplier for identification of optimal value of parameters in a functional. The VIM is computable and efficient for obtaining the solutions of linear and non-linear partial differential equations. As an application of this method, we solve some test problems and their solutions are represented graphically with the help of mathematica software. The results show that the proposed method has accelerated convergence.

Keywords: Linear and non linear partial differential equation, Lagrange multiplier, Variational Iteration Method, Mathematica.

1 Introduction

Linear and nonlinear partial differential equations play an important role in various fields of science and technology, because they describe many non-linear phenomenon. Nonlinear problems arise in different areas including gravitation, chemical reaction, fluid dynamics, dispersion, nonlinear optics, plasma physics and others. Non linear Diffusion equation have provided solutions of different physical phenomenon than solutions of linear Diffusion equation.

The Variational iteration method is first proposed by Ji-Huan He [1-4] which gives rapidly convergent successive approximations of the exact solution, if such solution exist. This method is preferable over numerical methods because it is free from rounding off errors and neither requires large computer power or memory. The VIM is first applied to obtain the analytical solutions of autonomous ordinary differential equations, non-linear partial differential equations with variable coefficients and integro-differential equations by He et. al. [1],[3],[4],[6],[7],[8]. Generally, one or two iterations lead to highly accurate solutions. This method is in fact, a modification of general Lagrange multiplier method in to an iteration method. Applications of the method have been enlarged due to its flexibility, convenience and efficiency. Recently, there has been great development in the numerical analysis. Numerical and analytical methods have included Adomian Decomposition Method, Variational Iteration

Method and Homotopy Perturbation method .VIM does not require specific treatment for nonlinear problem as in ADM , Perturbation etc.Considerable attention has been given to the variational iteration method for solving Telegraph Equations.The Telegraph equation arises in the propagation of electrical signals along a telegraph line[9].

We organize this paper as follows: In section 2, the basic definitions of Telegraph equation is given. Section 3, is devoted to develop the Variational iteration method for partial differential equations. In the last section, we solve some test problems and represent them graphically.

2 Telegraph Equation

In this section, we study about Telegraph equations.

Definition 2.1 *The initial boundary value Telegraph equation in its standard form is given by*

$$\begin{aligned} u_{xx}(x, t) &= au_{tt}(x, t) + bu_t(x, t) + cu(x, t) \\ \text{boundary conditions : } &u(0, t) = f(t), u_x(0, t) = g(t) \\ \text{initial conditions : } &u(x, 0) = h(x), u_t(x, 0) = v(x) \end{aligned}$$

Where $u(x,t)$ is the resistance and a,b,c are constants related to the inductance , capacitance and conductance of the cable respectively.

In the next section, we develop the Variational iteration method for Telegraph partial differential equation.

3 Variational Iteration Method for Telegraph Equation

To illustrate the basic idea of this method , we consider a general initial boundary value Telegraph equation as follows

$$u_{xx}(x, t) = au_{tt}(x, t) + bu_t(x, t) + cu(x, t) \quad (3.1)$$

$$\begin{aligned} \text{boundary conditions : } &u(0, t) = f(t), u_x(0, t) = g(t) \\ \text{initial conditions : } &u(x, 0) = h(x), u_t(x, 0) = v(x) \end{aligned}$$

The basic structure of He's VIM is the construction of correction functional for (3.1) is given by

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\xi) \left[a \frac{\partial^2 u_n(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x, \xi)}{\partial x^2} + b \frac{\partial u_n(x, \xi)}{\partial \xi} + c \tilde{u}_n(x, \xi) \right] d\xi. \quad (3.2)$$

where λ is general Lagrange multiplier which can be identified optimally via, variational theory and \tilde{u}_n is restricted variation which means $\delta\tilde{u}_n = 0$. Solving equation (3.2), we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_n(x, t), n > 0$ of solution $u(x, t)$ will be rapidly obtained using the obtained Lagrange multiplier and any selective function $u_0(x, t)$, consequently, the solution is

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) \quad (3.3)$$

Finally, we have the following variational iteration formula for the equation (3.1),

$$\begin{cases} u(x, 0) &= h(x) + tv(x) \quad \text{is initial guess,} \\ u_{n+1}(x, t) &= u_n(x, t) + \int_0^t \lambda(\xi) \left[a \frac{\partial^2 u_n(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x, \xi)}{\partial x^2} + b \frac{\partial u_n(x, \xi)}{\partial \xi} + c \tilde{u}_n(x, \xi) \right] d\xi. \end{cases} \quad (3.4)$$

where Lagrange multiplier λ is to be identify. In the next section, we develop the VIM for homogeneous Telegraph partial differential equation.

4 Applications:

Test Problem (i): Consider the homogeneous Telegraph partial differential equation

$$\begin{aligned} u_{xx}(x, t) &= u_{tt}(x, t) + u_t(x, t) + u \\ \text{boundary conditions : } &u(0, t) = e^{-t}, u_x(0, t) = e^{-t} \\ \text{initial conditions : } &u(x, 0) = e^x, u_t(x, 0) = -e^x \end{aligned}$$

The correctional functional for given equation is approximately expressed as follows

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\xi) \left[\frac{\partial^2 u_n(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x, \xi)}{\partial x^2} + \frac{\partial u_n(x, \xi)}{\partial \xi} + \tilde{u}_n(x, \xi) \right] d\xi.$$

The stationary conditions give

$$\lambda = \xi - t$$

Substituting the value of the Lagrange multiplier into the correction functional, we get the following iteration formula

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t (\xi - t) \left[\frac{\partial^2 u_n(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x, \xi)}{\partial x^2} + \frac{\partial u_n(x, \xi)}{\partial \xi} + \tilde{u}_n(x, \xi) \right] d\xi.$$

According to VIM we can select $u_0(x, t) = (1 - t)e^x$ from the given initial condition. Using this selection we obtain the successive approximations as follows :

$$\begin{aligned}
 u_0(x, t) &= (1 - t)e^x \\
 u_1(x, t) &= u_0(x, t) + \int_0^t (\xi - t) \left[\frac{\partial^2 u_0(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_0(x, \xi)}{\partial x^2} + \frac{\partial u_0(x, \xi)}{\partial \xi} + \tilde{u}_0(x, \xi) \right] d\xi. \\
 u_1(x, t) &= \left(1 - t + \frac{t^2}{2}\right)e^x \\
 u_2(x, t) &= \left(1 - t + \frac{t^2}{2} + \frac{t^3}{3!}\right)e^x \\
 &\vdots \\
 u_n(x, t) &= \left(1 - t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} - \dots\right)e^x
 \end{aligned}$$

This gives the exact solution by

$$u(x, t) = e^{x-t}$$

The graphical representation of the solution is as follow:

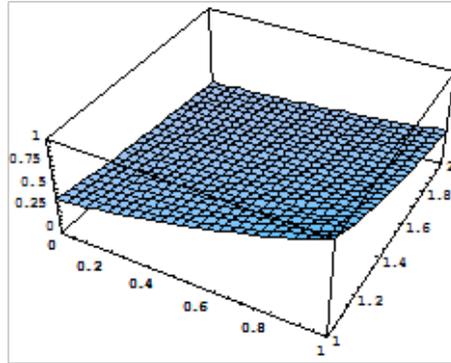


Fig.4.1 : The exact solution of the homogeneous Telegraph PDE

Test Problem (ii): Consider the homogeneous Telegraph partial differential equation

$$\begin{aligned}
 u_{xx}(x, t) &= u_{tt}(x, t) + 2u_t(x, t) + u(x, t) \\
 \text{boundary conditions : } &u(0, t) = e^{-t}, u_x(0, t) = 1 \\
 \text{initial conditions : } &u(x, 0) = 1 + \sinh x, u_t(x, 0) = -1
 \end{aligned}$$

The correctional functional for given equation is approximately expressed as follows

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\xi) \left[\frac{\partial^2 u_n(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x, \xi)}{\partial x^2} + 2 \frac{\partial u_n(x, \xi)}{\partial \xi} + \tilde{u}_n(x, \xi) \right] d\xi.$$

The stationary conditions give

$$\lambda = \xi - t$$

Substituting the value of the Lagrange multiplier into the correction functional, we get the following iteration formula

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t (\xi - t) \left[\frac{\partial^2 u_n(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x, \xi)}{\partial x^2} + 2 \frac{\partial u_n(x, \xi)}{\partial \xi} + \tilde{u}_n(x, \xi) \right] d\xi.$$

According to VIM we can select $u_0(x, t) = 1 + \sinh x - t$ from the given initial conditions. Using this selection we obtain the successive approximations as follows :

$$u_0(x, t) = 1 + \sinh x - t$$

$$u_1(x, t) = u_0(x, t) + \int_0^t (\xi - t) \left[\frac{\partial^2 u_0(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_0(x, \xi)}{\partial x^2} + 2 \frac{\partial u_0(x, \xi)}{\partial \xi} + \tilde{u}_0(x, \xi) \right] d\xi.$$

$$u_1(x, t) = (1 + \sinh x - t + \int_0^t (\xi - t)(-1 - \xi)$$

$$u_1(x, t) = \sinh x + 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}$$

$$u_2(x, t) = \sinh x + 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!}$$

⋮

$$u_n(x, t) = \sinh x + (1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \dots)$$

This gives the exact solution by

$$u(x, t) = \sinh x + e^{-t}$$

The graphical representation of the solution is as follow:

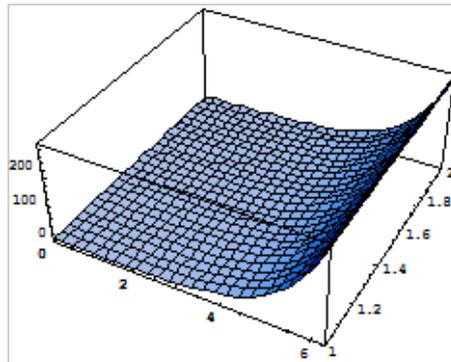


Fig.4.2 : The exact solution of the homogeneous Telegraph PDE

Test Problem (iii): Consider the homogeneous Telegraph partial differential equation

$$\begin{aligned} u_{xx}(x, t) &= u_{tt}(x, t) + u_t(x, t) + u \\ \text{boundary conditions : } &u(0, t) = 1 + e^{-t}, u_x(0, t) = 1 \\ \text{initial conditions : } &u(x, 0) = 1 + e^x, u_t(x, 0) = -1 \end{aligned}$$

The correctional functional for given equation is approximately expressed as follows

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\xi) \left[\frac{\partial^2 u_n(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x, \xi)}{\partial x^2} + \frac{\partial u_n(x, \xi)}{\partial \xi} + \tilde{u}_n(x, \xi) \right] d\xi.$$

The stationary conditions give

$$\lambda = \xi - t$$

Substituting the value of the Lagrange multiplier into the correction functional, we get the following iteration formula

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t (\xi - t) \left[\frac{\partial^2 u_n(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x, \xi)}{\partial x^2} + \frac{\partial u_n(x, \xi)}{\partial \xi} + \tilde{u}_n(x, \xi) \right] d\xi.$$

According to VIM we can select $u_0(x, t) = 1 + e^x - t$ from the given initial condition. Using this selection we obtain the successive approximations as follows :

$$\begin{aligned} u_0(x, t) &= 1 + e^x - t \\ u_1(x, t) &= u_0(x, t) + \int_0^t (\xi - t) \left[\frac{\partial^2 u_0(x, \xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_0(x, \xi)}{\partial x^2} + \frac{\partial u_0(x, \xi)}{\partial \xi} + \tilde{u}_0(x, \xi) \right] d\xi. \\ &= \left(1 - t + \frac{t^2}{2} - \frac{t^3}{3!} \right) + e^x \\ u_1(x, t) &= \left(1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} - \dots \right) + e^x \\ &\vdots \\ u_n(x, t) &= \left(1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} - \dots \right) + e^x \end{aligned}$$

This gives the exact solution by

$$u(x, t) = e^x + e^{-t}$$

The graphical representation of the solution is as follow:

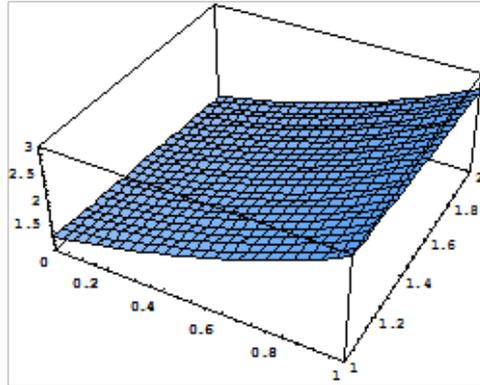


Fig.4.3 : The exact solution of the homogeneous Telegraph PDE

Conclusions

The main objective of this work is to obtain a solution for Telegraph partial differential equations. We observe that VIM is a powerful method to solve Telegraph partial differential equations. To, show the applicability and efficiency of the proposed method ,the method is applied to obtain the solutions of several examples. The obtained results demonstrate the reliability of the algorithm. It is worth mentioning that the proposed technique is capable of reducing the volume of the computational work as compared to the classical methods. Finally, we come to the conclusion that the vim is very powerful and efficient in finding solutions for wide class of Telegraph partial differential equations.

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