A Note on Maximal Ideals in Ternary Semigroups

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Abstract

In a commutative ring with identity element any maximal ideal is prime. Similar results hold in a commutative ordered semigroup with identity element ([2], [5]). In this note, we follow the idea in [2] by showing that the result holds on a commutative ternary semigroup with identity element. Moreover, we prove by an example that the converse of the statement does not hold, in general.

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1 Introduction and Preliminaries

There are a number of relations between maximal and prime ideals in a commutative ring with identity element. Such as, in a commutative ring with identity element any maximal ideal is prime. In 1969, Šchwarz proved the similar result on a semigroup, in [5, Theorem 1, p. 73]. In 2003, Kehayopulu, Ponizovskii and Tsingelis generalized the Šchwarz’ result, they have done on any ordered semigroups, in [2, p. 34]. In this paper, we follow the idea of [2] by showing that in any commutative ternary semigroup with identity element any maximal ideal is prime.

Ternary algebraic systems, a nonempty set with a ternary operation, have been introduced by Lehmer in 1932 [3] and studied by Siomon in 1965 [6]. Ternary semigroups were first introduced by Banach, he showed by an example that a ternary semigroup does not necessary reduce to an ordinary semigroup.

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Definition 1.1 Let $S$ be a nonempty set. Then $S$ is called a ternary semigroup if there exists a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, such that

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in S$.

Let $(S, \cdot)$ be a semigroup. For $x_1, x_2, x_3 \in S$, define a ternary operation on $S$ by $[x_1, x_2, x_3] = x_1 \cdot x_2 \cdot x_3$. Then $S$ is a ternary semigroup. The notion of ternary semigroups have been widely studied.

For nonempty subsets $A_1, A_2$ and $A_3$ of a ternary semigroup $S$, let

$$[A_1A_2A_3] = \{[x_1x_2x_3] \mid x_i \in A_i\}.$$  

For $x \in S$, let $[xA_1A_2] = \{[x]A_1A_2\}$. For any other cases can be defined analogously.

Definition 1.2 A ternary semigroup $S$ is said to be commutative if for any bijection $\alpha$ on $\{1, 2, 3\}$, $[x_1x_2x_3] = [x_{\alpha(1)}x_{\alpha(2)}x_{\alpha(3)}]$ for all $x_1, x_2, x_3 \in S$.

Definition 1.3 Let $(S, [, ,])$ be a ternary semigroup. A nonempty subset $A$ of $S$ is called an ideal of $S$ if

$$[ASS] \subseteq A, [SAS] \subseteq A \text{ and } [SSA] \subseteq A.$$  

In [1], p. 14, the intersection of all ideal of a ternary semigroup $S$ containing a nonempty subset $A$ of $S$ is an ideal of $S$. It is called the ideal of $S$ generated by $A$, denoted by $I(A)$. In a ternary semigroup $S$, $I(A)$ is of the form:

$$I(A) = A \cup [SSA] \cup [ASS] \cup [S[SSA]S].$$  

(1)

Definition 1.4 Let $(S, [, ,])$ be a ternary semigroup. An ideal $A$ of $S$ is said to be prime if for any $x, y \in S$, $[xSy] \subseteq A$ implies $x \in A$ or $y \in A$.

Definition 1.5 An ideal $A$ of a ternary semigroup $S$ is called a maximal ideal of $S$ if for any ideal $T$ of $S$, $A \subseteq T \subseteq S$ implies $T = A$ or $T = S$.

Let $\{(S_i, [, ,]) \mid i \in I\}$ be a nonempty family of ternary semigroups. Consider the Cartesian product $\prod_{i \in I} S_i$ of ternary semigroups $S_i$ for all $i \in I$. Define a ternary operation

$$\prod_{i \in I} S_i \times \prod_{i \in I} S_i \times \prod_{i \in I} S_i \to \prod_{i \in I} S_i,$$

written as

$$(x_i)_{i \in I}, (y_i)_{i \in I}, (z_i)_{i \in I} \mapsto [(x_i)_{i \in I}(y_i)_{i \in I}(z_i)_{i \in I}],$$

by

$$[(x_i)_{i \in I}(y_i)_{i \in I}(z_i)_{i \in I}] = ([x_iy_i]z_i)_{i \in I}.$$  

Then $\prod_{i \in I} S_i$ is a ternary semigroup.
2 Main Results

The following lemmas are required.

Lemma 2.1 Let \( \{(S_i, [\cdot, \cdot, \cdot]) \mid i \in I\} \) be a nonempty family of ternary semigroups. If \( A_i \) is an ideal of \( S_i \) for each \( i \in I \), then the set \( \prod_{i \in I} A_i \) is an ideal of \( \prod_{i \in I} S_i \).

Proof. Since \( A_i \neq \emptyset \) for all \( i \in I \), there exists \( x_i \in A_i \) for each \( i \in I \). Since \( (x_i)_{i \in I} \in \prod_{i \in I} A_i \), \( \prod_{i \in I} A_i \neq \emptyset \).

Let \( (x_i)_{i \in I} \in \prod_{i \in I} A_i \) and \( (y_i)_{i \in I}, (z_i)_{i \in I} \in \prod_{i \in I} S_i \). Since

\[
[x_i y_i z_i] \in [A_i S_i S_i]_i \subseteq A_i
\]

for every \( i \in I \), it follows that

\[
[(x_i)_{i \in I}(y_i)_{i \in I}(z_i)_{i \in I}] = ([x_i y_i z_i]_{i \in I} \in \prod_{i \in I} A_i.
\]

Then \( [(\prod_{i \in I} A_i)(\prod_{i \in I} S_i)(\prod_{i \in I} S_i)] \subseteq \prod_{i \in I} A_i \). In the same manner, we have that

\[
[(\prod_{i \in I} S_i)(\prod_{i \in I} S_i)(\prod_{i \in I} A_i)] \subseteq \prod_{i \in I} A_i
\]

and

\[
[(\prod_{i \in I} S_i)(\prod_{i \in I} A_i)(\prod_{i \in I} S_i)] \subseteq \prod_{i \in I} A_i.
\]

Therefore, the set \( \prod_{i \in I} A_i \) is an ideal of \( \prod_{i \in I} S_i \).

Let \( S = [0, 1] \) be the closed interval of real numbers. Using the usual multiplication, \( S \) is a ternary semigroup under the ternary operation defined by

\[
[x_1 x_2 x_3] = x_1 \cdot x_2 \cdot x_3
\]

for all \( x_1, x_2, x_3 \in S \). Next, we show that any closed interval \([0, a] \) where \( a \in S \) is an ideal of \( S \).

Lemma 2.2 If \( a \in S \), then the set \( A_a = [0, a] \) is an ideal of \( S \).

Proof. Let \( a \in S \). Since \( a \in [0, a] \), \( A_a \neq \emptyset \). We shall show that \( [A_a SS] \subseteq A_a \). Let \( x \in A_a \) and \( y, z \in S \). Since \( 0 \leq x \leq a, 0 \leq y, z \leq 1 \), we obtain \( 0 \leq xyz \leq a \). Thus \( [xyz] \in A_a \). Then \( [A_a SS] \subseteq A_a \). Similarly, \( [SSA_a] \subseteq A_a \) and \( [SA_a S] \subseteq A_a \). Therefore, \( A_a \) is an ideal of \( S \).

An element \( e \) of a ternary semigroup \( S \) is called an identity element of \( S \) if

\[
[exx] = [xex] = [xxe] = x
\]

for all \( x \in S \).

The following theorem shows that in a commutative ternary semigroup with identity element any maximal ideal is prime.
Theorem 2.3 Let \((S,[,])\) be a commutative ternary semigroup with identity element. If \(M\) is a maximal ideal of \(S\), then \(M\) is prime.

Proof. Let \(e\) be an identity element of \(S\). Assume that \(M\) is a maximal ideal of \(S\). To show that \(M\) is a prime ideal of \(S\), let \(x, y \in M\) be such that \([xSy] \subseteq M\) and \(x \notin M\). Since \(S\) is commutative, by (1), we have

\[
I(M \cup \{x\}) = (M \cup \{x\}) \cup [(M \cup \{x\})SS] \cup [SS(M \cup \{x\})]
\]

\[
\cup [S(S(M \cup \{x\})S)]
\]

\[
= (M \cup \{x\}) \cup [SS(M \cup \{x\})].
\]

Since \(M \cup \{x\} = [ee(M \cup \{x\})] \subseteq [SS(M \cup \{x\})]\), it follows that

\[
I(M \cup \{x\}) = [SS(M \cup \{x\})].
\] (2)

Since \(x \notin M\),

\[
M \subseteq M \cup \{x\} \subseteq I(M \cup \{x\}).
\]

Since \(M\) is maximal we obtain \(I(M \cup \{x\}) = S\). By (2), \(e \in [SS(M \cup \{x\})]\). Let \(z, w \in S\) and \(t \in M \cup \{x\}\) such that \(e = [zwt]\). Then

\[
y = [yey] = [yzwt]y.
\]

If \(t \in M\), then

\[
[yzwt]y = [yzwty] \in [SMS] \subseteq M,
\]

so \(y \in M\). If \(t = x\), because \(S\) is commutative, then

\[
[yzwt]y = [yzwxy] = [yzxwy] = [ywy] \in [SM] \subseteq M.
\]

So \(y \in M\). Therefore, \(M\) is a prime ideal of \(S\).

Remark The converse of Theorem 2.3 does not holds, in general. The following example shows. Let \(S = [0,1]\) be a ternary semigroup mentioned above. We consider the ternary semigroup \(S \times S\). Clearly, \(S \times S\) is commutative and has the identity \((1,1)\). Let

\[
T = S \times \{0\}(\{0,1\} \times \{0\}).
\]

By Lemma 2.2, \(A_0 = \{0\}\) is an ideal of \(S\). Since \(S\) is an ideal of \(S\), by Lemma 2.1, we obtain \(T\) is an ideal of \(S \times S\).

Let \((x, y), (z, w) \in S \times S\) be such that \([(x, y)S \times S(z, w)] \subseteq T\). Then \([(x1z), [y1w]] \in T\). Since \(yw = [y1w] = 0\), \(y = 0\) or \(w = 0\). This implies that \((x, y) \in T\) or \((z, w) \in T\). Thus \(T\) is a prime ideal of \(S \times S\). By Lemma 2.2, \(A_2 = [0,\frac{1}{2}]\) is an ideal of \(S\). Since \(T = S \times \{0\} \subset S \times A_2 \subset S \times S\) we have that \(T\) is not maximal.
References


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