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Optimal control of stochastic differential delay equations with application in economics

Anatoli F. Ivanov[a,*], Anatoly V. Swishchuk[b]

[a]*Department of Mathematics, Pennsylvania State University, P.O. Box PSU, Lehman, PA 18627, USA*
[b]*Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta T2N 1N4, Canada*

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Abstract. The paper is devoted to the study of optimal control of stochastic differential delay equations and their applications. By using the Dynkin formula and solution of the Dirichlet-Poisson problem, the Hamilton-Jacobi-Bellman (HJB) equation and the inverse HJB equation are derived. Application is given to a stochastic model in economics.

AMS Subject Classifications: 34K50, 34K35, 34K60, 91B70

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1. Introduction

In our presentation at the Conference on Stochastic Modelling of Complex Systems SMOCS05 [8] the following controlled stochastic differential delay equation (SDDE) was introduced:

$$x(t) = x(0) + \int_0^t a(x(s-1), u(s))ds + \int_0^t b(x(s-1), u(s))dw(s),$$

where $x(t) = \phi(t), \quad t \in [-1, 0]$, is a given continuous process, $u(t)$ is a control process and $w(t)$ is a standard Wiener process.

E-mail addresses: afi1@psu.edu (A. F. Ivanov), aswish@ucalgary.ca (A. V. Swishchuk)
*Corresponding author

We presented Dynkin formula and solution of Dirichlet-Poisson problem for the SDDE. These results can be obtained from the relevant results about the Dynkin formulas and boundary value problems for multiplicative operator functionals of Markov processes [13]. By using the Dynkin formula and the solution of the Dirichlet-Poisson problem, the Hamilton-Jacobi-Bellman (HJB) and the inverse HJB equations have been stated. We have also found the stochastic optimal control and optimal performance for the SDDE. The results there have been presented without proof.

In present paper, we give a complete proof of two theorems from the talk: Theorem 1 (HJB equation) and Theorem 2 (inverse of the HJB equation) about the stochastic optimal control. For the definitions related to the stochastic optimal control and stochastic optimal performance see [11]. Application is given to a stochastic model in economics, a Ramsey model [4, 12] that takes into account the delay and randomness in the production cycle.

The Ramsey model is described by the equation

$$dK(t) = [AK(t-T) - u(K(t))C(t)]\,dt + \sigma(K(t-T))dw(t)$$

where K is the capital, C is the production rate, u is a control process, A is a positive constant, σ is a standard deviation of the "noise". The "initial capital"

$$K(t) = \phi(t), \quad t \in [-T, 0],$$

is a continuous bounded positive function. For this stochastic economic model the optimal control is found to be $u_{\min} = K(0) \cdot C(0)$, and the optimal performance is

$$\begin{aligned} J(K, u_{\min}) &= \tfrac{K^2(0)}{2} + \tfrac{K^2(0)\cdot C^2(0)}{2} + \int_{-T}^{0} \phi^2(\theta)\,d\theta \\ &= \tfrac{K^2(0)}{2}(1 + C^2(0)) + \int_{-T}^{0} \phi^2(\theta)\,d\theta. \end{aligned}$$

By time rescaling, the delay T can be normalized to $T = 1$, which will be our assumption in the theoretical considerations that follow. The obtained results are valid however for general delay $T > 0$.

2. Controlled Stochastic Differential Delay Equations

2.1. Assumptions and existence of solutions

Below we recall some basic notions and facts from [3, 6, 7, 10] necessary for subsequent exposition in this paper. Let $x(t), t \in [-1, \infty)$ be a stochastic process, $\mathcal{F}_{\alpha\beta}(x)$ be a minimal σ-algebra with respect to which $x(t)$ is measurable for every $t \in [\alpha, \beta]$. Let $w(t), t \in [-1, \infty)$ be a Wiener process with $w(0) = 0$, and let $\mathcal{F}_{\alpha\beta}(dw)$ be a minimal Borel σ-algebra such that $w(t) - w(s)$ is measurable for all t, s with $\alpha \le t \le s \le \beta$. Let $u(t) \in \mathcal{U}, t \in [-1, \infty)$ be a stochastic process whose values can be chosen from the given Borel set $\mathcal{U}$ and such that $u(t)$ is $\mathcal{F}_{\alpha\beta}(u)$-adapted for all $t \in [\alpha, \beta]$.

Let $\mathcal{C}$ denote the metric space of all continuous functions defined on the interval $[-1, 0]$ with the standard norm $|h| = \sup_{-1 \le t \le 0} |h(t)|$. One also has the notation

$h_t(s) := h(t+s), s \in [-1,0]$. If $h(t)$ is continuous for $t \geq -1$ then $h_t \in \mathcal{C}$. For definitions, notations, and basics of the deterministic differential delay equations see e.g. [5].

Let $a(\cdot,\cdot), b(\cdot,\cdot)$ be continuous functionals defined on $\mathcal{C} \times \mathcal{U}$. A stochastic process $x(t)$ is called a solution of the stochastic differential delay equation

$$dx(t) = a(x_t, u(t))dt + b(x_t, u(t))dw(t), \quad t \in [0,\infty) \tag{2.1}$$

if

$$\mathcal{F}_{-1t}(x) \vee \mathcal{F}_{0t}(dw) \vee \mathcal{F}_{0t}(u)$$

is independent of $\mathcal{F}_{t\infty}(dw)$ for every $t \in [0,\infty)$. Here $\mathcal{F}_{-1t}(x) \vee \mathcal{F}_{0t}(dw) \vee \mathcal{F}_{0t}(u)$ stands for the minimal σ-algebra containing $\mathcal{F}_{-1t}(x)$, $\mathcal{F}_{0t}(dw)$, and $\mathcal{F}_{0t}(u)$, and

$$x(t) - x(s) = \int_s^t a(x_r, u(r))dr + \int_s^t b(x_r, u(r))dw(r),$$

where the last integral is the Ito integral.

Equation (2.1) is meant in the integral form

$$x(t) = x(0) + \int_0^t a(x_s, u(s))ds + \int_0^t b(x_s, u(s))dw(s) \tag{2.2}$$

with the initial condition $x(t) = \phi(t), t \in [-1,0]$, where $\phi \in \mathcal{C}$ is a given continuous function. Therefore, we assume that the processes $\phi(t), t \in [-1,0], w(t)$ and $u(t), t \geq 0$, are defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $\mathcal{F}_t := \mathcal{F}_{-1t}(x) \vee \mathcal{F}_{0t}(dw) \vee \mathcal{F}_{0t}(u)$.

Let the following conditions be satisfied for equation (2.2).

A.1 $a(\phi, u)$ and $b(\phi, u)$ are continuous real-valued functionals defined on $\mathcal{C} \times \mathcal{U}$;

A.2 $\phi \in \mathcal{C}$ is continuous with probability 1 in the interval $[-1,0]$, independent of $w(s), s \geq 0$, and $E|\phi(t)|^4 < \infty$;

A.3 $\forall \phi, \psi \in \mathcal{C}$:

$$|a(\phi,u) - a(\psi,u)| + |b(\phi,u) - b(\psi,u)| \leq K \int_{-1}^0 |\phi(\theta) - \psi(\theta)| d\theta, \tag{2.3}$$

with $|a(0,u)| + |b(0,u)| \leq M$ for some $M, K > 0$ and all $u \in \mathcal{U}$.

Under assumptions A.1-A.3 the solution $x(t)$ of the initial value problem (2.2) exists and is a unique stochastic continuous process [3, 6, 10]. The function x_t is a Markov process. The solution can be viewed at time $t \geq 0$ as an element x_t of the space $\mathcal{C}$, or as a point $x(t)$ in $\mathbb{R}$. We shall use both interpretations in this paper, as appropriate.

2.2. Weak infinitesimal operator of Markov process $(x_t, x(t))$

In the case of stochastic differential delay equations the solution $x(t)$ is not Markovian. However, we can Markovianize it by considering the pair $(x_t, x(t)) := (x(t+s), x(t))$, $s \in [-1, 0]$, i.e., the path of the process from $t-1$ till t and the value of the process at t. The pair is a strong Markov process to which we can apply the weak infinitesimal generator (see e.g. [2]).

A real valued functional $J(x_t, x(t))$ on $\mathcal{C} \times \mathbb{R}$ is said to be in the *domain* of A^u, the weak infinitesimal operator (w.i.o.), if the limit

$$\lim_{t\to 0+}((E^u_{x,x(0)} J(x_t, x(t)) - J(x, x(0)))/t) = q(x, x(0), u),$$
$$x = x_0 = \phi \in \mathcal{C},\ u \in \mathcal{U}$$

exists pointwise in $\mathcal{C} \times \mathcal{U}$, and

$$\lim_{t\to 0+} \sup_{x,u} |E^u_{x,x(0)} q(x_t, x(t), u) - q(x, x(0), u)| = 0.$$

Here $x_t := x_t(\theta) = x(t+\theta), \theta \in [-1, 0]$, is in $\mathcal{C}$ and E^u_x is the expectation under the conditional probability with respect to x and u. We set $A^u J(x, x(0)) := q(x, x(0), u)$.

For an open and bounded set $H \times G \subset \mathcal{C} \times \mathbb{R}$ denote by $\tilde{A}^u_{H\times G}$ the w.i.o. of $(\tilde{x}_t, \tilde{x}(t)) := (x_t, x(t))$ stopped at $\tau_{H\times G} := \inf\{t : (x_t, x(t)) \notin H \times G\}$ [2].

Let $F : \mathbb{R} \to \mathbb{R}$ be continuous and bounded on bounded sets and set $J(x_t, x(t)) := F(x(t))$. Then if $F \in D(\tilde{A}^u_G)$ and $\tilde{A}^u_G F = q$ is bounded on bounded sets, the restriction of F to G is in $D(\tilde{A}^u_G)$, and

$$\begin{aligned} \tilde{A}^u_G J(x) &= L^u F(x(0)) = q(x(0), u) := \\ &= F'(x(0)) a(x(0), u) + F''(x(0)) \tfrac{1}{2} b^2(x(0), u) \end{aligned} \tag{2.4}$$

where $u = u(0)$ (see[10]).

It is not simple to completely characterize the domain of the weak infinitesimal operator of either processes ϕ or $x(t)$. For example, in the case of $J(x) = x(-1)$ the operator is not necessarily in $D(\tilde{A}^u_G)$, since $x(t)$ can be not differentiable.

It is possible to study functionals $J(x(0))$ whose dependence on $\phi \in \mathcal{C}$ is in the form of an integral. For example, let the above conditions be satisfied for the functional

$$J_\phi(x(0)) := \int_{-1}^{0} F(\phi(s), x(0))\, ds,$$

where $F : \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ is continuous. Let in addition $F(\phi, x)$, $F'_x(\phi, x)$, $F''_{xx}(\phi, x)$ be continuous in ϕ, x. Then $J_\phi(x) \in D(\tilde{A}^u_G)$ and

$$\begin{aligned} \tilde{A}^u_G J_\phi(x(0)) = q(x(0)) &= F(\phi(0), x(0)) - F(\phi(-1), x(0)) + \\ &+ \int_{-1}^{0} L^u F(\phi(s), x(0))\, ds, \end{aligned} \tag{2.5}$$

where the operator L^u is defined by (2.4) and acts on F as a function of $x(0)$ only (see [10]).

2.3. Dynkin formula for SDDEs

Let $x(t)$ be a solution of the initial value problem (2.2). For the strong Markov process $(x_t, x(t))$ consider the functional

$$J(x_t, x(t)) := \int_{-1}^{0} F(x(t+\theta), x(t))\, d\theta.$$

From (2.5) we obtain the following Ito formula for the functional J :

$$\begin{aligned} J(x_t, x(t)) =& J(x, x(0)) + \int_0^t F(x(s), x(s))\, ds - \\ & - \int_0^t F(x(s-1), x(s))\, ds + \int_0^t \int_{-1}^0 L^u F(x(t+\theta), x(s))\, d\theta ds + \\ & + \int_0^t \int_{-1}^0 \sigma(x(s-1)) F'_x(x(t+\theta), x(s))\, d\theta dw(s). \end{aligned}$$

Let τ be a stopping time for the strong Markov process $(x_t, x(t))$ such that $E_{x,x(0)}|\tau| < \infty$. Then we have the following Dynkin formula [13]

$$\begin{aligned} E_{x,x(0)} J(x_\tau, x(\tau)) =& J(x, x(0)) + E_{x,x(0)} \int_0^\tau F(x(s), x(s))\, ds \\ & - \int_0^\tau F(x(s-1), x(s))\, ds \\ & + E_{x,x(0)} \int_0^\tau \int_{-1}^0 L^u F(x_s(\theta), x(s))\, d\theta ds \\ =& J(x, x(0)) + E_{x,x(0)} \int_0^\tau \tilde{A}^u_G J(x_s, x(s))\, ds, \end{aligned} \tag{2.6}$$

where $\tilde{A}^u_G$ is defined by (2.5).

2.4. Solution of Dirichlet-Poisson problem for SDDEs

Let $G \subset \mathbb{R}$ and $H \subset \mathcal{C}$ be bounded open sets, and $\partial(H \times G)$ be the regular boundary of the set $H \times G$. Let $\psi(x, x(0))$ be a given function continuous on the closure of the set $H \times G$ and bounded on $\partial(H \times G)$. Let function $F(x, x(0), u) \in C(\mathcal{C} \times \mathbb{R} \times \mathcal{U})$ be such that

$$E_{x,x(0)} \left[\int_{-1}^0 \int_0^{\tau_{H\times G}} |F(\phi(\theta), x(s), u(s))|\, ds d\theta \right] < \infty \qquad \forall\ (x, x(0)) \in H \times G,$$

where $\tau_{H\times G} = \inf\{t : (x_t, x(t)) \notin H \times G\}$ is the exit time from the set $H \times G$.

Define

$$\begin{aligned} J(x, x(0), u) :=& E_{x,x(0)} \left[\int_{-1}^0 \int_0^{\tau_{H\times G}} F(x_s(\theta), x(s), u(s))\, ds d\theta \right] \\ & + E_{x,x(0)} \left[\psi(x_{\tau_{H\times G}}, x(\tau_{H\times G})) \right], \qquad (x, x(0)) \in H \times G. \end{aligned}$$

Then [13]

$$\tilde{A}^u J(x, x(0), u) = -\int_{-1}^{0} F(\phi(\theta), x, u)\, d\theta, \qquad \text{in} \quad H \times G \quad \forall u \in \mathcal{U}$$

and

$$\lim_{t \uparrow \tau_{H\times G}} J(x_t, x(t), u) = \psi(x_{\tau_{H\times G}}, x(\tau_{H\times G})) \qquad \forall (x, x(0)) \in H \times G.$$

2.5. Statement of the Problem

We assume that the cost function is given in the form

$$\begin{aligned} J(x, x(0), u) := & E_{x,x(0)} \Bigg[\int_{-1}^{0} \int_{0}^{\tau_{H\times G}} F(x_s(\theta), x(s), u(s))\, ds d\theta \\ & + \psi(x_{\tau_{H\times G}}, x(\tau_{H\times G})) \Bigg], \end{aligned} \tag{2.7}$$

where ψ, F and $\tau_{H\times G}$ are as in subsection 2.4. In particular, $\tau_{H\times G}$ can be a fixed time t_0. We assume that $E_{x,x(0)}|\tau_{H\times G}| < \infty, \forall (x, x(0)) \in H \times G$. Similar cost functions are considered in [11] for systems without the delay.

The problem is as follows. For each $(x, x(0)) \in H \times G$ find a number $J^*(x, x(0))$ and a control $u^* = u^*(x, x(0)), \omega)$ such that

$$J^*(x, x(0)) := \inf_u \{J(x, x(0), u)\} = J(x, x(0), u^*),$$

where the infinum is taken over all $\mathcal{F}_t$-adapted processes $u(t) \in \mathcal{U}$. Such a control u^*, if it exists, is called an *optimal control* and $J^*(x, x(0))$ is called the *optimal performance*.

3. Hamilton-Jacobi-Bellman Equation for SDDEs

We consider only Markov controls $u(t) := u(x_t, x(t))$. For every $\nu \in \mathcal{U}$ define the following operator

$$\begin{aligned} (A^\nu J)(x, x(0)) = & F(x(0), x(0), \nu(0)) - F(x(-1), x(0), \nu(0)) \\ & + \int_{-1}^{0} L^\nu F(\phi(\theta), x(0), \nu(0))\, d\theta, \quad \nu(0) := \nu(x, x(0)), \end{aligned} \tag{3.1}$$

where operator L^ν is given by (2.4), and let

$$J(x, x(0)) := \int_{-1}^{0} F(\phi(\theta), x(0), \nu(0))\, d\theta.$$

With $x(t)$ being the solution of equation (2.2), for each control u the pair $(x_t, x(t))$ is an Ito diffusion with the infinitesimal generator $(AJ)(x, x(0)) = (A^u J)(x, x(0))$.

Theorem 3.1. (HJB-equation) *Let*

$$J^*(x, x(0)) = \inf\{J(x, x(0), u) | u := u(x, x(0)) \text{ - } \textit{Markov control}\}. \tag{3.2}$$

Suppose that $J \in C^2(H \times G)$ *and the optimal control* u^* *exists. Then*

$$\inf_{\nu \in \mathcal{U}} \left[\int_{-1}^{0} F(\phi(\theta), x, \nu)\, d\theta + (A^{\nu} J^*)(x, x(0)) \right] = 0, \quad \forall (x, x(0)) \in H \times G, \tag{3.3}$$

and

$$J^*(x, x(0)) = \psi(x, x(0)), \quad \forall (x, x(0)) \in \partial(H \times G),$$

where F *and* ψ *are as in (2.7), and operator* A^{ν} *is given by (3.1).*

The infinum in (3.2) is achieved when $\nu = u^*(x, x(0))$, *where* u^* *is optimal. In other words,*

$$\int_{-1}^{0} F(\phi(\theta), x, u^*)\, d\theta + (A^{u^*} J^*)(x, x(0)) = 0, \quad \forall (x, x(0)) \in H \times G, \tag{3.4}$$

which is equation (3.3).

Proof. Now we proceed to prove (3.3). Fix $(x, x(0)) \in H \times G$ and choose a Markov control process u. Let $\alpha \leq \tau_{H \times G}$ be a stopping time. By using the strong Markov property of $(x_t, x(t))$ we obtain for $J(x, x(0), u)$:

$$\begin{aligned}
&E_{x,x(0)}[J(x_\alpha, x(\alpha), u)] \\
&\quad = E_{x,x(0)}\Bigg[E_{x_\alpha, x(\alpha)}\bigg[\int_{-1}^{0} \int_{0}^{\tau_{H\times G}} F(x_s(\theta), x(s), u(s))\, ds d\theta \\
&\qquad + \psi(x_{\tau_{H\times G}}, x(\tau_{H\times G})) \bigg]\Bigg] \\
&\quad = E_{x,x(0)}\Bigg[E_{x,x(0)}\bigg[S_\alpha \bigg(\int_{-1}^{0} \int_{0}^{\tau_{H\times G}} F(x_s(\theta), x(s), u(s))\, ds d\theta \\
&\qquad + \psi(x_{\tau_{H\times G}}, x(\tau_{H\times G})) \bigg) / F_\alpha \bigg]\Bigg] \\
&\quad = E_{x,x(0)}\Bigg[E_{x,x(0)}\bigg[\int_{-1}^{0} \int_{0}^{\tau_{H\times G}} F(x_s(\theta), x(s), u(s))\, ds d\theta \\
&\qquad + \psi(x_{\tau_{H\times G}}, x(\tau_{H\times G})) \bigg] / F_\alpha \Bigg] \\
&\quad = E_{x,x(0)}\bigg[\int_{-1}^{0} \int_{0}^{\tau_{H\times G}} F(x_s(\theta), x(s), u(s))\, ds d\theta \\
&\qquad + \psi(x_{\tau_{H\times G}}, x(\tau_{H\times G})) - \int_{-1}^{0} \int_{0}^{\alpha} F(x_s(\theta), x(s), u(s))\, ds d\theta \bigg] \\
&\quad = J(x, x(0), u) - E_{x,x(0)}\bigg[\int_{-1}^{0} \int_{0}^{\alpha} F(x_s(\theta), x(s), u(s))\, ds d\theta \bigg],
\end{aligned}$$

where S_α is a shift operator (see, e.g., [2]). Therefore

$$\begin{aligned} J(x, x(0), u) =& E_{x,x(0)}\left[\int_{-1}^{0}\int_{0}^{\alpha} F(x_s(\theta), x(s), u(s))\, ds d\theta\right] \\ &+ E_{x,x(0)}\left[J(x_\alpha, x(\alpha), u(\alpha))\right]. \end{aligned} \tag{3.5}$$

Now let $V \subset H\times G$ be of the form $V := \{(y, y(0)) \in H\times G : |(y, y(0))-(x, x(0))| < \epsilon\}$. Let $\alpha = \tau_V$ be the first exit time of the pair $(x_t, x(t))$ from V.

Suppose the optimal control u^* exists. For every $\nu \in \mathcal{U}$ choose:

$$u = \begin{cases} \nu(x, x(0)), & \text{if} \quad (x, x(0)) \in V \\ u^*(x, x(0)), & \text{if} \quad (x, x(0)) \in H \times G \setminus V. \end{cases} \tag{3.6}$$

Then $J^*(x_\alpha, x(\alpha)) = J(u^*, x_\alpha, x(\alpha))$, and by combining (3.5) and(3.6), we obtain

$$\begin{aligned} J^*(x, x(0)) \le J(x, x(0), \nu) =& E_{x,x(0)}\left[\int_{-1}^{0}\int_{0}^{\alpha} F(x_s(\theta), x(r), \nu(r))\, d\theta dr\right] \\ &+ E_{x,x(0)}\left[J(x_\alpha, x(\alpha), \nu)\right]. \end{aligned}$$

By Dynkin formula (2.6) we have

$$E_{x,x(0)}\left[J(x_\alpha, x(\alpha), \nu)\right] = J(x, x(0)) + E_{x,x(0)}\left[\int_{0}^{\alpha} A^\nu J(x_r, x(r), \nu)\, dr\right],$$

where A^ν is defined by (3.1). By substituting the latter into the previous inequality we obtain

$$\begin{aligned} J^*(x, x(0)) \le& E_{x,x(0)}\left[\int_{-1}^{0}\int_{0}^{\alpha} F(x_s(\theta), x(s), \nu(s))\, ds\right] + J(x, x(0)) \\ &+ E_{x,x(0)}\left[\int_{0}^{\alpha} A^\nu J(x_r, x(r), \nu)\, dr\right], \end{aligned}$$

or

$$E_{x,x(0)}\left[\int_{-1}^{0}\int_{0}^{\alpha} F(x_r(\theta), x(r), \nu(r))\, dr d\theta + \int_{0}^{\alpha} A^\nu J(x_r, x(r), \nu)\, dr\right] \ge 0.$$

Therefore,

$$\begin{aligned} E_{x,x(0)}&\left[\int_{-1}^{0}\int_{0}^{\alpha} F(x_r(\theta), x(r), \nu(r))\, d\theta dr + \right. \\ &\left. + \int_{0}^{\alpha} (A^\nu J)(x_r, x(r), \nu)\, dr\right] / E_{x,x(0)}[\alpha] \ge 0. \end{aligned}$$

By letting $\epsilon \to 0$ we derive

$$\int_{-1}^{0} F(x, x(0), \nu)\, d\theta + (A^\nu J)(x, x(0), \nu) \ge 0,$$

which combined with (3.4) gives (3.3). $\square$

Theorem 3.2. (Converse of the HJB-equation) *Let g be a bounded function in $C^2(H \times G) \cap C(\partial(H \times G))$. Suppose that for all $u \in \mathcal{U}$ the inequality*

$$\int_{-1}^{0} F(x, x(0), u)\, d\theta + (A^u g)(x, x(0)) \geq 0, \quad (x, x(0)) \in H \times G$$

and the boundary condition

$$g(x, x(0)) = \psi(x, x(0)), \quad (x, x(0)) \in \partial(H \times G) \tag{3.7}$$

are satisfied. Then $g(x, x(0)) \leq J(x, x(0), u)$ for all Markov controls $u \in \mathcal{U}$ and for all $(x, x(0)) \in H \times G$.

Moreover, if for every $(x, x(0)) \in H \times G$ there exists u^0 such that

$$\int_{-1}^{0} F(x, x(0), u^0)\, d\theta + (A^{u^0} g)(x, x(0)) = 0, \tag{3.8}$$

then u^0 is a Markov control, $g(x) = J(x, x(0), u^0) = J^*(x, x(0))$, and therefore u^0 is an optimal control.

Proof. Assume that g satisfies hypotheses (3.7) and (3.8). Let u be a Markov control. Then $A^u J \geq -\int_{-1}^{0} F(x, x(0), u)\, d\theta$ for all u in $\mathcal{U}$, and we have by Dynkin formula (2.6)

$$\begin{aligned} E_{x,x(0)}\left[g(x_{\tau_r}, x(\tau_r))\right] &= g(x, x(0)) + E_{x,x(0)}\left[\int_0^{\tau_r} (A^u g)(x_s, x(s))\, ds\right] \\ &\geq g(x, x(0)) - E_{x,x(0)}\left[\int_{-1}^{0}\int_0^{\tau_r} F(x_s(\theta), x(s), u(s))\, d\theta ds\right], \end{aligned}$$

where

$$\tau_r := \min\{r, \tau_{H \times G}, \inf\{t > 0 : |x_t| \geq r\}\}, \qquad r > 0.$$

By taking the limit as $\tau_r \to +\infty$ this gives

$$\begin{aligned} g(x, x(0)) &\leq E_{x,x(0)}\left[\int_{-1}^{0}\int_0^{\tau_r} F(x_s(\theta), x(s), u(s))\, d\theta ds + \psi(x_{\tau_r}, x(\tau_r))\right] \\ &\leq \lim_{\tau_r \to \infty}\left[E_{x,x(0)}\left[\int_{-1}^{0}\int_0^{\tau_r} F(x_s(\theta), x(s), u(s))\, d\theta ds + \psi(x_{\tau_r}, x(\tau_r))\right]\right] \\ &= E_{x,x(0)}\left[\int_{-1}^{0}\int_0^{\tau_{H\times G}} F(x_s(\theta), x(s), u(s))\, d\theta ds + \psi(x_{\tau_{H\times G}}, x(\tau_{H\times G}))\right] \\ &= J(x, x(0), u), \end{aligned}$$

which proves the first assertion of the theorem. If u^0 is such that (3.8) holds, then the above calculation gives the equality. This completes the proof. $\square$

Remark 3.1. The Hamilton-Jacobi-Bellman (HJB) equation and the Inverse HJB equation are classical results in the optimization theory. Theorems 1 and 2 above provide their extensions to the case of stochastic differential delay equations considered in this paper. Both statements assume the existence of the optimal control u_0 as a hypothesis. The existence of the optimal control in a general setting is an important and difficult problem by itself. Under certain conditions on functions a, b, F, ϕ and the boundary of the set $H \times G$, and with the compactness of the set of control values, one can show, by using related general results from nonlinear PDEs, that a smooth function J satisfying equation (3.3) and boundary condition $J^*(x, x(0)) = \phi(x, x(0))$ exists. Then by applying a measurable selection theorem one can find a measurable function u^* satisfying equation (3.4) (or equation (3.8)) for almost all points in $H \times G$. For more details of this possible approach to tackle the existence problem see for example [1, 9]. We plan to address this general problem of existence of optimal control in our future research. In the next section we show the existence of the optimal control and find it explicitly for the Ramsey SDDE Model with a given cost function.

4. Economics Model and Its Optimization

4.1. Description of the model

In 1928 F.R. Ramsey introduced an economics model describing the rate of change of capital K and labor L in a market by a system of ordinary differential equations [12]. With P and C being the production and consumption rates, respectively, the model is given by the system

$$\frac{dK(t)}{dt} = P(t) - C(t), \qquad \frac{dL(t)}{dt} = a(t)L(t), \tag{4.1}$$

where $a(t)$ is the rate of growth of labor (population).

The production, capital and labor are related by the Cobb-Douglas formula, $P(t) = A\,K^{\alpha}(t)\,L^{\beta}(t)$, where A, α, β are some positive constants [4]. In certain circumstances the dependence of P on K and L is linear, i.e. $\alpha = \beta = 1$, which will be our assumption throughout this section. We shall also assume that the labor is constant, $L(t) = L_0$, which is true for certain markets or relatively short time intervals of several years. Therefore, the production rate and the capital are related by $P(t) = B\,K(t)$, where $B = A\,L_0$. Another important assumption we make is that the production rate is subject to small random disturbances, i.e. $P(t) = B\,K(t) +$ "noise". System (4.1) then results in the equation

$$\frac{dK(t)}{dt} = B\,K(t) + \text{"noise"} - C(t),$$

which can be rewritten in the differential form as

$$dK(t) = [B\,K(t) - C(t)]\,dt + \sigma(K(t))\,dw(t),$$

where $w(t)$ is a standard Wiener process, $\sigma(K)$ is a given (small) real function, characteristic of the noise.

The original model of Ramsey is based on the assumption of instant transformation of the investments. This can be accepted as satisfactory in only very rough models. In the reality the trasformation of the invested capital cannot be accomplished instantly. A certain essential period of time is normally required for this transformation, such as the length of the production cycle in many economical situations. Therefore, a more accurate assumption is that the rate of change of capital K at present time t depends on the investment that was made at time $t - T$, where T is the cycle duration required for the creation of working capital. This leads to the following delay differential equation

$$dK(t) = [B\,K(t-T) - C(t)]\,dt + \sigma(K(t-T))\,dw(t).$$

Our next assumption is that the consumption rate C can be controlled by the available amount of the capital, i.e. it is of the form $C(t)\,u(K(t))$, where $u(\cdot)$ is a control. By normalizing the delay to $T = 1$ (by time rescaling) one arrives at the equation

$$dK(t) = [B\,K(t-1) - u(K(t))\,C(t)]\,dt + \sigma(K(t-1))\,dw(t). \tag{4.2}$$

The initial investment of the capital K is naturally represented in equation (4.2) by a given initial function ϕ

$$K(t) = \phi(t), \quad t \in [-1, 0]. \tag{4.3}$$

Therefore, we propose to study a modified Ramsey model with delay and random perturbations given by the system (4.2)-(4.3).

4.2. Optimization calculation

Usually one wants to minimize the investment capital under the assumption of labor being constant. Let us choose the following cost function

$$J(K,u) = \frac{K^2(0)}{2} + \int_{-1}^{0} \phi^2(\theta)\,d\theta + \frac{u^2(0)}{2}.$$

The operator $A^u J$ has the following form

$$\begin{aligned} A^u J = &\frac{K^2(0)}{2} + \phi^2(0) + \frac{u^2(0)}{2} - \left[\frac{K^2(0)}{2} + \phi^2(-1) + \frac{u^2(0)}{2}\right] \\ &+ \left[K(0)\cdot(B\cdot K(0) - u(0)\cdot C(0)) + \frac{1}{2}\sigma^2(K(0))\right], \end{aligned}$$

since

$$\begin{aligned} F(K(0),K(0),u(0)) &= \frac{K^2(0)}{2} + \phi^2(0) + \frac{u^2(0)}{2}, \\ F(K(0),K(-1),u(0)) &= \frac{K^2(0)}{2} + \phi^2(-1) + \frac{u^2(0)}{2}, \\ L^u J(K,u) &= K(0)\left(B\cdot K(0) - u(0)\cdot C(0) + \frac{1}{2}\sigma^2(K(0))\right). \end{aligned}$$

From (3.1) we obtain the following HJB-equation

$$\inf_u \left[\frac{K^2(0)}{2} + \int_{-1}^{0} \phi^2(\theta)\, d\theta + \frac{u^2(0)}{2} + \phi^2(0) - \phi^2(-1) + B \cdot K^2(0) \right.$$
$$\left. + \; u(0) \cdot K(0) \cdot C(0) + \frac{1}{2}\sigma^2(K(0))\right] = 0,$$

or equivalently,

$$\inf_u \Bigg[u^2(0) - 2K(0)C(0)u(0) + (2\phi^2(0) - 2\phi^2(-1) + 2\int_{-1}^{0} \phi^2(\theta)\, d\theta$$
$$+ K^2(0)(1+2B) + \sigma^2(K(0)))\Bigg] = 0.$$

Let

$$4K^2(0)C^2(0) \geq 4\Bigg(2\phi^2(0) - 2\phi^2(-1) + 2\int_{-1}^{0} \phi^2(\theta)\, d\theta$$
$$+ K^2(0)(1+2B) + \sigma^2(K(0))\Bigg),$$

or

$$K^2(0) \cdot (C^2(0) - 3 - 2B) \geq 2\int_{-1}^{0} \phi^2(\theta)\, d\theta - 2\phi^2(-1) + \sigma^2(K(0)),$$

since $K(0) = \phi(0)$. Hence, the infinum is achieved when

$$u(0) = -\left(-\frac{2K(0) \cdot C(0)}{2}\right) = K(0) \cdot C(0).$$

Therefore $u_{\min} = K(0) \cdot C(0)$ and

$$J(K, u_{\min}) = \frac{K^2(0)}{2} + \frac{K^2(0) \cdot C^2(0)}{2} + \int_{-1}^{0} \phi^2(\theta)\, d\theta =$$
$$= \frac{K^2(0)}{2}(1 + C^2(0)) + \int_{-1}^{0} \phi^2(\theta)\, d\theta.$$

Note that in the case of general delay $T > 0$ in model (4.2)-(4.3) the last expression for J remains valid with the integration range $[-T, 0]$.

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