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Existence of positive solution for a class of fourth-order singular m-point boundary value problems*

Meiqiang Feng[a,c,†], Hanying Feng[b,c], Weigao Ge[c]

[a] *Department of Mathematics, Beijing Information Science & Technology University, Beijing, 100192, PR China*
[b] *Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, PR China*
[c] *Department of Applied Mathematics, Beijing Institute of Technology, Beijing, 100081, PR China*

Abstract. The authors consider the existence of positive solutions for a class of fourth-order nonlinear singular m-point boundary value problem with p-Laplacian by using fixed point theory in cones, and derive an explicit interval for λ such that for any λ in this interval, the existence of at least one positive solution to the boundary value problem is guaranteed. The results significantly extend and improve many known results even for non-singular cases.

AMS Subject Classifications: 34B15

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1. Introduction

Recently, there have been many papers working on the existence of positive solutions to multi-point boundary-value problems for ordinary differential equations, see, for example, [1,3,8-16]. This has been mainly due to its arising in different areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform

E-mail addresses: meiqiangfeng@sina.com (M. Feng), fhanying@yahoo.com.cn (H. Feng), gew@bit.edu.cn (W. Ge)
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†Corresponding author

cross-section and composed of N parts of different densities can be set up as a multi-point boundary-value problem, many problems in the theory of elastic stability can be handled as multi-point boundary-value problems too. The singular ordinary differential equations arise in the fields of gas dynamics, Newtonian fluid mechanics, the theory of boundary layer and so on, the theory of singular boundary value problems has become an important area of investigation in recent years. To identify a few, we refer the reader to [3,4,16,18] and references therein.

Motivated by works mentioned above, in this paper, we study the existence of positive solutions for fourth-order singular m-point boundary value problem (BVP) with p-Laplacian

$$\begin{cases} (\phi_p(x''))'' - \lambda g(t)f(t,x) = 0, \qquad 0 < t < 1, \\ x''(0) = x''(1) = 0, \\ ax(0) - bx'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i), \\ cx(1) + dx'(1) = \sum_{i=1}^{m-2} b_i x(\xi_i), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\phi_p(s)$ is p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s, p > 1, (\phi_p)^{-1} = \phi_q, \frac{1}{p} + \frac{1}{q} = 1$, $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, $\rho := ac+bc+ad > 0$, $\xi_i \in (0,1)$, $a_i, b_i \in (0,+\infty)$ $(i = 1,2,\dots,m-2)$, $g \in C((0,1),[0,+\infty))$ and $0 < \int_0^1 g(t)dt < +\infty$, $f \in C([0,1] \times [0,+\infty),[0,+\infty))$.

However, to the best of our knowledge, there are very few literatures considering the existence of positive solutions for fourth-order singular m-point boundary value problem, see, for example, [4-7]. But the boundary conditions are two-point in [4-7]. So it is interesting and important to discuss the existence of positive solutions for problem (1.1). Many difficulties occur when we deal with them. For example, the construction of cone and operator. So we need to introduce some new tools and methods to investigate the existence of positive solutions for problem (1.1). On the other hand, $g(t)$ may be singular at $t = 0$ and/or $t = 1$. Moreover, the methods used in this paper are different from [4,5] and the results obtained in this paper generalize some results in [4-7] to some degree.

The paper is organized in the following fashion. In Section 2, we provide some necessary background. In particular, we state some properties of the Green's function associated with BVP (2.2), BVP (2.7) and some lemmas that are important to our main results. In Section 3, the main result will be stated and proved.

2. Preliminaries

Let $J = [0,1]$. The basic space used in this paper is $E = C[0,1]$. It is well known that E is a real Banach space with the norm $\|\cdot\|$ defined by $\|x\| = \max_{t\in J} |x(t)|$.

The following assumptions will stand throughout this paper:

(H_1) $g \in C((0,1),[0,+\infty))$ and $0 < \int_0^1 g(t)dt < +\infty$;

(H_2) $f \in C([0,1] \times [0,+\infty),[0,+\infty))$;

(H_3) $\Delta < 0$, $\rho - \sum_{i=1}^{m-2} a_i\phi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) > 0$, where

$$\Delta = \begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i\psi(\xi_i) & \rho - \sum\limits_{i=1}^{m-2} a_i\phi(\xi_i) \\ \rho - \sum\limits_{i=1}^{m-2} b_i\psi(\xi_i) & -\sum\limits_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix},$$

and

$$\psi(t) = b + at, \qquad \phi(t) = c + d - ct, \qquad t \in J \tag{2.1}$$

are linearly independent solutions of the equation $x'' = 0$.

We remark that (2.1) shows that ψ is nondecreasing on J and ϕ is nonincreasing on J.

In our main results, we will make use of the following lemmas.

Lemma 2.1. *If (H_1) and (H_2) hold, then BVP*

$$\begin{cases} -y''(t) = -\lambda g(t)f(t,x(t)), & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{2.2}$$

has a unique solution

$$y(t) = -\lambda \int_0^1 h(t,s)g(s)f(s,x(s))ds, \tag{2.3}$$

where

$$h(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{2.4}$$

By calculation, it is easy to prove that (2.3) holds.

It is clear that $h(t,s)$ has the following properties.

Proposition 2.1. *For $t, s \in J$, we have*

$$0 \le h(t,s) \le h(s,s) \le \frac{1}{4}. \tag{2.5}$$

Proposition 2.2. *Let $\theta \in (0, \frac{1}{2})$ and define $J_\theta = [\theta, 1-\theta]$. Then for all $t \in J_\theta$, $s \in [0,1]$ we have*

$$h(t,s) \ge \theta h(s,s). \tag{2.6}$$

Proof. Let $t \in J_\theta$, $s \in [0,1]$. We distinguish two cases:
Case 1: $\theta \le t \le s \le 1$

In this case, we have

$$h(t,s) = t(1-s) = \frac{t}{s} h(s,s) \ge t h(s,s) \ge \theta h(s,s).$$

Case 2: $0 \le s \le t \le 1-\theta$

In this case, we have

$$h(t,s) = s(1-t) = \frac{1-t}{1-s} h(s,s) \ge (1-t) h(s,s) \ge \theta h(s,s).$$

Therefore, (2.6) holds. $\square$

Lemma 2.2. *If* (H_3) *holds, then, for* $y \in C[0,1]$*, BVP*

$$\begin{cases} -x'' = -\phi_q(y(t)), & 0 < t < 1, \\ ax(0) - bx'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i), \\ cx(1) + dx'(1) = \sum_{i=1}^{m-2} b_i x(\xi_i), \end{cases} \tag{2.7}$$

has a unique solution

$$x(t) = -\left[\int_0^1 G(t,s)\phi_q(y(s))ds + A(\phi_q(y))\psi(t) + B(\phi_q(y))\phi(t) \right], \tag{2.8}$$

where

$$G(t,s) = \tfrac{1}{\rho} \begin{cases} \psi(s)\phi(t), & \textit{if } \ 0 \le s \le t \le 1, \\ \psi(t)\phi(s), & \textit{if } \ 0 \le t \le s \le 1, \end{cases} \tag{2.9}$$

$$A(\phi_q(y)) := \frac{1}{\Delta} \begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)\phi_q(y(t))dt & \rho - \sum_{i=1}^{m-2} a_i \phi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)\phi_q(y(t))dt & -\sum_{i=1}^{m-2} b_i \phi(\xi_i) \end{vmatrix}, \tag{2.10}$$

$$B(\phi_q(y)) := \frac{1}{\Delta} \begin{vmatrix} -\sum_{i=1}^{m-2} a_i \psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)\phi_q(y(t))dt \\ \rho - \sum_{i=1}^{m-2} b_i \psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)\phi_q(y(t))dt \end{vmatrix}. \tag{2.11}$$

The proof is similar to that of Lemma 5.5.1 in [9].
Let

$$\nabla := \min\{\min_{t\in J_\theta}\phi(t), \min_{t\in J_\theta}\psi(t), 1\}, \qquad \Lambda := \max\{1, \|\psi\|, \|\phi\|\}.$$

It is not difficult to show that $G(t,s)$ have the following properties.

Proposition 2.3. *For $t, s \in J$, we have*

$$0 \le G(t,s) \le G(s,s) \le \frac{\Lambda^2}{\rho}. \tag{2.12}$$

Proposition 2.4. *For $t \in J_\theta$, $s \in [0,1]$ we have*

$$G(t,s) \ge \sigma G(s,s), \tag{2.13}$$

where

$$\sigma = \sigma_\theta = \min\left\{\tfrac{\psi(\theta)}{\psi(1)}, \tfrac{\phi(1-\theta)}{\phi(0)}\right\}. \tag{2.14}$$

Proof. Let $t \in J_\theta$, $s \in [0,1]$. We distinguish two cases:
Case 1: $\theta \le t \le s \le 1$
In this case, we have

$$G(t,s) = \frac{1}{\rho}\psi(t)\phi(s) = \frac{\psi(t)}{\psi(s)}G(s,s) \ge \frac{\psi(\theta)}{\psi(1)}G(s,s).$$

Case 2: $0 \le s \le t \le 1-\theta$
In this case, we have

$$G(t,s) = \frac{1}{\rho}\phi(t)\psi(s) = \frac{\phi(t)}{\phi(s)}G(s,s) \ge \frac{\phi(1-\theta)}{\phi(0)}G(s,s).$$

Therefore, (2.13) holds. □

Suppose that $x(t)$ is a solution of (1.1). Then from Lemma 2.1 and Lemma 2.2, we have

$$x(t) = \lambda^{q-1}\left[\int_0^1 G(t,s)\phi_q(\Upsilon)ds + A(\phi_q(\Upsilon))\psi(t) + B(\phi_q(\Upsilon))\phi(t)\right], \tag{2.15}$$

where $\Upsilon = \int_0^1 h(s,\tau)g(\tau)f(\tau, x(\tau))d\tau$.

Similar to the proof of that Lemma 5.5.2 and Lemma 5.5.3 in [10] we can prove that the following Lemma holds.

Lemma 2.3. *Suppose that $(H_1)-(H_3)$ hold. Then the solution x of (1.1) satisfies $x(t) \ge 0$ for $t \in J$ and $\min_{t\in J_\theta} x(t) \ge \bar\sigma\|x\|$, where $\bar\sigma = \min\{\sigma, \frac{\nabla}{\Lambda}\}$.*

Let

$$K = \{x \in E : x \geq 0, \min_{t \in J_\theta} x(t) \geq \bar{\sigma}\|x\|\}.$$

It is clear that K is a cone of E.

Define an operator $T : K \to K$ by

$$(Tx)(t) = \lambda^{q-1}\left[\int_0^1 G(t,s)\phi_q(\Upsilon)ds + A(\phi_q(\Upsilon))\psi(t) + B(\phi_q(\Upsilon))\phi(t)\right]. \tag{2.16}$$

Lemma 2.4. *Suppose that $(H_1) - (H_3)$ hold. Then $TK \subset K$ and $T : K \to K$ is completely continuous.*

Proof. For any $x \in K$, by (2.16), we obtain $T_\lambda x \geq 0$ and

$$\begin{aligned}(Tx)(t) &= \lambda^{q-1}[\textstyle\int_0^1 G(t,s)\phi_q(\Upsilon)ds + A(\phi_q(\Upsilon))\psi(t) + B(\phi_q(\Upsilon))\phi(t)]\\ &\leq \lambda^{q-1}\left[\textstyle\int_0^1 G(s,s)\phi_q(\Upsilon)ds + \Lambda[A(\phi_q(\Upsilon)) + B(\phi_q(\Upsilon))]\right], \ for\ t \in J.\end{aligned}$$

On the other hand, we have for $t \in J_\theta$

$$\begin{aligned}(Tx)(t) &= \lambda^{q-1}[\textstyle\int_0^1 G(t,s)\phi_q(\Upsilon)ds + A(\phi_q(\Upsilon))\psi(t) + B(\phi_q(\Upsilon))\phi(t)]\\ &\geq \lambda^{q-1}\left[\sigma \textstyle\int_0^1 G(s,s)\phi_q(\Upsilon)ds + \frac{\nabla}{\Lambda}\Lambda[A(\phi_q(\Upsilon)) + B(\phi_q(\Upsilon))]\right]\\ &\geq \bar{\sigma}\lambda^{q-1}\left[\textstyle\int_0^1 G(s,s)\phi_q(\Upsilon)ds + \Lambda[A(\phi_q(\Upsilon)) + B(\phi_q(\Upsilon))]\right]\\ &\geq \bar{\sigma}\|Tx\|.\end{aligned}$$

Therefore, $TK \subset K$.

Next by standard methods and Ascoli-Arzela theorem one can prove $T : K \to K$ is completely continuous. So it is omitted. □

Lemma 2.5. ([17] Fixed point theorem of cone expansion and compression of norm type) *Let Ω_1 and Ω_2 be two bounded open sets in Banach space E, such that $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let operator $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be completely continuous, where θ denotes the zero element of E and P is a cone in E. Suppose that one of the two conditions*

(i) $||Ax|| \leq ||x||, \forall x \in P \cap \partial\Omega_1$ *and* $||Ax|| \geq ||x||, \forall x \in P \cap \partial\Omega_2$,

or

(ii) $||Ax|| \geq ||x||, \forall x \in P \cap \partial\Omega_1,$ *and* $||Ax|| \leq ||x||, \forall x \in P \cap \partial\Omega_2$,

is satisfied. Then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

3. Main results

We are now ready to apply the Guo-Krasnoselskii fixed point theorem to the operator T to give sufficient conditions for the existence of positive solution to BVP (1.1).

Let

$$\min f_\infty := \liminf_{x\to\infty} \min_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)}, \qquad \max f_0 := \limsup_{x\to 0^+} \max_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)},$$

$$\min f_0 := \liminf_{x\to 0^+} \min_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)}, \qquad \max f_\infty := \limsup_{x\to \infty} \max_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)}.$$

Theorem 3.1. *Assume that* $(H_1)-(H_3)$ *hold, and* $\min f_\infty > 0$, $\max f_0 < \infty$. *Then BVP (1.1) has at least one positive solution in* P *provided*

$$\frac{1}{\bar\sigma^{p-1}\min f_\infty \cdot L_1^{p-1}} < \lambda < \frac{1}{\max f_0 \cdot M_1^{p-1}}, \tag{3.1}$$

where

$$M_1 = \left(\frac{1}{4}\right)^{q-1}\left[\frac{\Lambda^2}{\rho}\phi_q\left(\int_0^1 g(\tau)d\tau\right) + \Lambda[\tilde A + \tilde B]\right],$$

$$L_1 = (\theta)^{2(q-1)}(1-\theta)^{q-1}\left[\frac{1}{\rho}\sigma\psi(\theta)\phi(1-\theta)\phi_q\left(\int_\theta^{1-\theta} g(\tau)d\tau\right) + \nabla[\hat A + \hat B]\right],$$

where $\tilde A, \tilde B$ *are defined by (3.4) and (3.5),* $\hat A, \hat B$ *are defined by (3.8) and (3.9), respectively.*

Proof. Let T be cone preserving, completely continuous operator that was defined by (2.16).

By (3.1), there exists $\varepsilon > 0$ such that

$$\frac{1}{\bar\sigma^{p-1}(\min f_\infty - \varepsilon) \cdot L_1^{p-1}} \le \lambda \le \frac{1}{(\max f_0 + \varepsilon) \cdot M_1^{p-1}}. \tag{3.2}$$

Considering $\max f_0 < \infty$, there exists $r_1 > 0$ such that

$$f(t,x) \le (\max f_0 + \varepsilon)\phi_p(x) \qquad \text{for} \qquad 0 < x \le r_1, \quad t \in [0,1]. \tag{3.3}$$

So, for $x \in \partial P_{r_1}$, we have from (3.3)

$$\begin{aligned} |A(\phi_q(\Upsilon))| &\le \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)\phi_q\left(\int_0^1 g(\tau)d\tau\right)dt & \rho - \sum_{i=1}^{m-2} a_i\phi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)\phi_q\left(\int_0^1 g(\tau)d\tau\right)dt & -\sum_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix} \\ &\quad \times \left(\frac{1}{4}\right)^{q-1}(\max f_0 + \varepsilon)^{q-1}\|x\| \\ &:= \tilde A \left(\frac{1}{4}\right)^{q-1}(\max f_0 + \varepsilon)^{q-1}\|x\|, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
|B(\phi_q(\Upsilon))| \quad &\leq \quad \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)\phi_q\left(\int_0^1 g(\tau)d\tau\right)dt \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)\phi_q\left(\int_0^1 g(\tau)d\tau\right)dt \end{vmatrix} \\
&\quad \times\left(\frac{1}{4}\right)^{q-1}(\max f_0+\varepsilon)^{q-1}\|x\| \\
&:= \quad \tilde{B}\left(\frac{1}{4}\right)^{q-1}(\max f_0+\varepsilon)^{q-1}\|x\|. \qquad (3.5)
\end{aligned}$$

Therefore, by (3.3) − (3.5), for $x \in \partial P_{r_1}$ we have

$$\begin{aligned}
\|Tx\| &= \lambda^{q-1}\max_{t\in J}\left|\int_0^1 G(t,s)\phi_q(\Upsilon)ds + A(\phi_q(\Upsilon))\psi(t) + B(\phi_q(\Upsilon))\phi(t)\right| \\
&\leq (\lambda\tfrac{1}{4})^{q-1}(\max f_0+\varepsilon)^{q-1}\|x\|\left[\tfrac{\Lambda^2}{\rho}\phi_q(\textstyle\int_0^1 g(\tau)d\tau) + \max_{t\in J}[\tilde{A}\psi(t)+\tilde{B}\phi(t)]\right] \\
&\leq (\lambda\tfrac{1}{4})^{q-1}(\max f_0+\varepsilon)^{q-1}\|x\|\left(\tfrac{\Lambda^2}{\rho}\phi_q(\textstyle\int_0^1 g(\tau)d\tau) + \Lambda[\tilde{A}+\tilde{B}]\right) \\
&= \|x\|.
\end{aligned} \qquad (3.6)$$

Next, turning to $\min f_\infty > 0$, there exists $\bar{r}_2 > 0$ such that

$$f(t,x) \geq (\min f_\infty - \varepsilon)\phi_p(x) \quad \text{for } x \geq \bar{r}_2, \quad t\in[0,1]. \qquad (3.7)$$

Choose $r_2 = \max\{\frac{\bar{r}_2}{\bar{\sigma}}, r_1+1\}$, then $r_2 > r_1$. If $x \in \partial P_{r_2}$, then

$$\min_{t\in J_\theta} x(t) \geq \bar{\sigma}\|x\| = \bar{\sigma}r_2 \geq \bar{r}_2,$$

and we obtain by (3.7)

$$\begin{aligned}
|A(\phi_q(\Upsilon))| \quad &\geq \quad \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_\theta^{1-\theta} G(\xi_i,t)\phi_q\left(\int_\theta^{1-\theta} g(\tau)d\tau\right)dt & \rho-\sum_{i=1}^{m-2} a_i\phi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_\theta^{1-\theta} G(\xi_i,t)\phi_q\left(\int_\theta^{1-\theta} g(\tau)d\tau\right)dt & -\sum_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix} \\
&\quad \times(\theta)^{2(q-1)}(1-\theta)^{q-1}(\min f_\infty-\varepsilon)^{q-1}\bar{\sigma}\|x\| \\
&:= \quad \hat{A}(\theta)^{2(q-1)}(1-\theta)^{q-1}(\min f_\infty-\varepsilon)^{q-1}\bar{\sigma}\|x\|, \qquad (3.8)
\end{aligned}$$

and

$$
\begin{aligned}
|B(\phi_q(\Upsilon))| &\geq \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_\theta^{1-\theta} G(\xi_i,t)\phi_q\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big)dt \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_\theta^{1-\theta} G(\xi_i,t)\phi_q\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big)dt \end{vmatrix} \\
&\quad \times(\theta)^{2(q-1)}(1-\theta)^{q-1}(\min f_\infty-\varepsilon)^{q-1}\bar{\sigma}\|x\| \\
&:= \hat{B}(\theta)^{2(q-1)}(1-\theta)^{q-1}(\min f_\infty-\varepsilon)^{q-1}\bar{\sigma}\|x\|. \qquad (3.9)
\end{aligned}
$$

Therefore, by (2.16) and (3.7) $-$ (3.9), for $x \in \partial P_{r_2}$ we have

$$
\begin{aligned}
\|Tx\| &= \lambda^{q-1}\max_{t\in J}[\textstyle\int_0^1 G(t,s)\phi_q(\Upsilon)ds + A(\phi_q(\Upsilon))\psi(t)+B(\phi_q(\Upsilon))\phi(t)] \\
&\geq (\min f_\infty-\varepsilon)^{q-1}\lambda^{q-1}(\theta)^{2(q-1)}(1-\theta)^{q-1}\bar{\sigma}\|x\| \\
&\quad \times\left[\tfrac{1}{\rho}\sigma\psi(\theta)\phi(1-\theta)\phi_q(\textstyle\int_\theta^{1-\theta} g(\tau)d\tau)+\nabla[\hat{A}+\hat{B}]\right] \qquad (3.10)\\
&= \|x\|.
\end{aligned}
$$

Applying (i) of Lemma 2.5 to (3.6) and (3.10) yields that T has a fixed point $x^* \in \bar{P}_{r_1,r_2} = \{x : x \in P, r_1 \leq \|x\| \leq r_2\}$, $r_1 \leq \|x^*\| \leq r_2$ and $x^*(t) \geq \bar{\sigma}\|x^*\| > 0$, $t \in [0,1]$. Thus it follows that BVP (1.1) has a positive solution x^*. The proof is complete. □

Theorem 3.2. *Assume that* $(H_1)-(H_3)$ *hold, and* $\min f_0 > 0$, $\max f_\infty < \infty$. *Then BVP (1.1) has at least one positive solution in P provided*

$$\frac{1}{\bar{\sigma}^{p-1}\min f_0\cdot L_1^{p-1}} < \lambda < \frac{1}{\max f_\infty\cdot M_1^{p-1}}. \qquad (3.11)$$

Proof. Let T be cone preserving, completely continuous operator that was defined by (2.16).

By (3.11), there exists $\varepsilon > 0$ such that

$$\frac{1}{\bar{\sigma}^{p-1}(\min f_0-\varepsilon)\cdot L_1^{p-1}} \leq \lambda \leq \frac{1}{(\max f_\infty+\varepsilon)\cdot M_1^{p-1}}. \qquad (3.12)$$

Considering $\min f_0 > 0$, there exists $r_3 > 0$ such that

$$f(t,x) \geq (\min f_0-\varepsilon)\phi_p(x) \qquad \text{for } 0 \leq x \leq r_3,\quad t\in[0,1]. \qquad (3.13)$$

Then for $x \in \partial P_{r_3}$ we have by (3.13)

$$
\begin{aligned}
|A(\phi_q(\Upsilon))| &\geq \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_\theta^{1-\theta} G(\xi_i,t)\phi_q\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big)dt & \rho-\sum_{i=1}^{m-2} a_i\phi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_\theta^{1-\theta} G(\xi_i,t)\phi_q\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big)dt & -\sum_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix} \\
&\quad \times(\theta)^{2(q-1)}(1-\theta)^{q-1}(\min f_0-\varepsilon)^{q-1}\bar{\sigma}\|x\| \\
&:= \hat{A}(\theta)^{2(q-1)}(1-\theta)^{q-1}(\min f_0-\varepsilon)^{q-1}\bar{\sigma}\|x\|, \qquad (3.14)
\end{aligned}
$$

and

$$\begin{aligned}
|B(\phi_q(\Upsilon))| \quad &\geq \quad \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_\theta^{1-\theta} G(\xi_i,t)\phi_q\left(\int_\theta^{1-\theta} g(\tau)d\tau\right)dt \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_\theta^{1-\theta} G(\xi_i,t)\phi_q\left(\int_\theta^{1-\theta} g(\tau)d\tau\right)dt \end{vmatrix} \\
&\quad \times(\theta)^{2(q-1)}(1-\theta)^{q-1}(\min f_0-\varepsilon)^{q-1}\bar{\sigma}\|x\| \\
&:= \quad \hat{B}(\theta)^{2(q-1)}(1-\theta)^{q-1}(\min f_0-\varepsilon)^{q-1}\bar{\sigma}\|x\|. \qquad (3.15)
\end{aligned}$$

Therefore, by (2.16) and (3.13) − (3.15), for $x \in \partial P_{r_2}$ we have

$$\begin{aligned}
\|Tx\| \quad &= \quad \lambda^{q-1}\max_{t\in J}\left|\int_0^1 G(t,s)\phi_q(\Upsilon)ds + A(\phi_q(\Upsilon))\psi(t) + B(\phi_q(\Upsilon))\phi(t)\right| \\
&\geq \quad (\min f_0-\varepsilon)^{q-1}\lambda^{q-1}(\theta)^{2(q-1)}(1-\theta)^{q-1}\bar{\sigma}\|x\| \\
&\qquad \times\left[\tfrac{1}{\rho}\sigma\psi(\theta)\phi(1-\theta)\phi_q(\textstyle\int_\theta^{1-\theta} g(\tau)d\tau) + \nabla[\hat{A}+\hat{B}]\right] \qquad (3.16) \\
&= \quad \|x\|.
\end{aligned}$$

Next, turning to $\max f_\infty < \infty$, there exists $\bar{r}_4 > 0$ such that $f(t,x) \leq (\max f_\infty + \varepsilon)\phi_p(x)$ for $x \geq \bar{r}_4, \quad t \in [0,1]$.

Case 1) Suppose that f is bounded. Then there exists $R > 0$ such that $f(t,x) \leq \phi_p(R)$ for $t \in [0,1], \quad x \in [0,\infty)$. Choose

$$r_4 = \max\left\{2r_3, (\lambda\frac{1}{4})^{q-1}R\left(\frac{\Lambda^2}{\rho}\phi_q\left(\int_0^1 g(\tau)d\tau\right) + \Lambda[\tilde{A}+\tilde{B}]\right)\right\}.$$

Then for $x \in \partial P_{r_4}$ we have

$$\begin{aligned}
|A(\phi_q(\Upsilon))| \quad &\leq \quad \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)\phi_q\left(\int_0^1 g(\tau)d\tau\right)dt & \rho-\sum_{i=1}^{m-2} a_i\phi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)\phi_q\left(\int_0^1 g(\tau)d\tau\right)dt & -\sum_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix} \\
&\quad \times\left(\frac{1}{4}\right)^{q-1}R \\
&:= \quad \tilde{A}\left(\frac{1}{4}\right)^{q-1}R, \qquad (3.17)
\end{aligned}$$

and

$$\begin{aligned}
|B(\phi_q(\Upsilon))| &\leq \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)\phi_q\left(\int_0^1 g(\tau)d\tau\right)dt \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)\phi_q\left(\int_0^1 g(\tau)d\tau\right)dt \end{vmatrix} \\
&\quad \times \left(\frac{1}{4}\right)^{q-1} R \\
&:= \tilde{B}\left(\frac{1}{4}\right)^{q-1} R. \qquad (3.18)
\end{aligned}$$

Therefore, by (3.17) and (3.18), for $x \in \partial P_{r_4}$ we have

$$\begin{aligned}
\|Tx\| &= \max_{t\in J}\left[\int_0^1 G(t,s)\phi_q(\Upsilon)ds + A(\phi_q(\Upsilon))\psi(t) + B(\phi_q(\Upsilon))\phi(t)\right] \\
&\leq (\lambda\tfrac{1}{4})^{q-1}R\left[\tfrac{\Lambda^2}{\rho}\phi_q(\textstyle\int_0^1 g(\tau)d\tau) + \tilde{A}\psi(t) + \tilde{B}\phi(t)\right] \\
&\leq (\lambda\tfrac{1}{4})^{q-1}R\left(\tfrac{\Lambda^2}{\rho}\phi_q(\textstyle\int_0^1 g(\tau)d\tau) + \Lambda[\tilde{A}+\tilde{B}]\right) \\
&\leq r_4 \\
&= \|x\|.
\end{aligned} \qquad (3.19)$$

Case 2) Suppose that f is unbounded. Choosing $r_4 > \max\{2r_3, \bar{r}_4\}$ such that $f(t,x) \leq f(t,r_4)$ for $t \in [0,1]$, $x \in (0,r_4)$. Then for $x \in \partial P_{r_4}$ we have

$$\begin{aligned}
|A(\phi_q(\Upsilon))| &\leq \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)\phi_q\left(\int_0^1 g(\tau)d\tau\right)dt & \rho-\sum_{i=1}^{m-2} a_i\phi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)\phi_q\left(\int_0^1 g(\tau)d\tau\right)dt & -\sum_{i=1}^{m-2} b_i\phi(\xi_i) \end{vmatrix} \\
&\quad \times \left(\frac{1}{4}\right)^{q-1} (\max f_\infty + \varepsilon)^{q-1} r_4 \\
&:= \tilde{A}\left(\frac{1}{4}\right)^{q-1} (\max f_\infty + \varepsilon)^{q-1} r_4, \qquad (3.20)
\end{aligned}$$

and

$$
\begin{aligned}
|B(\phi_q(\Upsilon))| \quad &\leq \quad \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big)dt \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big)dt \end{vmatrix} \\
&\quad \times \left(\frac{1}{4}\right)^{q-1}(\max f_\infty+\varepsilon)^{q-1} r_4 \\
&:= \quad \tilde{B}\left(\frac{1}{4}\right)^{q-1}(\max f_\infty+\varepsilon)^{q-1} r_4. \qquad (3.21)
\end{aligned}
$$

Therefore, by (3.20) and (3.21), for $x \in \partial P_{r_4}$ we have

$$
\begin{aligned}
\|Tx\| \quad &= \quad \max_{t\in J}\left| \int_0^1 G(t,s)\phi_q(\Upsilon)ds + A(\phi_q(\Upsilon))\psi(t) + B(\phi_q(\Upsilon))\phi(t)\right| \\
&\leq \quad (\lambda\tfrac{1}{4})^{q-1}(\max f_\infty+\varepsilon)^{q-1} r_4\left[\tfrac{\Lambda^2}{\rho}\phi_q(\textstyle\int_0^1 g(\tau)d\tau) + \max_{t\in J}[\tilde{A}\psi(t)+\tilde{B}\phi(t)]\right] \\
&\leq \quad (\lambda\tfrac{1}{4})(\max f_\infty+\varepsilon)^{q-1} r_4\left(\tfrac{\Lambda^2}{\rho}\phi_q(\textstyle\int_0^1 g(\tau)d\tau) + \Lambda[\tilde{A}+\tilde{B}]\right) \\
&\leq \quad r_4 \\
&= \quad \|x\|.
\end{aligned}
\qquad (3.22)
$$

Applying (ii) of Lemma 2.5 to (3.16) and (3.19) or (3.22) yields that T has a fixed point $x^{**} \in \bar{P}_{r_3,r_4}$, $r_3 \leq \|x^{**}\| \leq r_4$ and $x^{**}(t) \geq \bar{\sigma}\|x^{**}\| > 0$, $t \in [0,1]$. Thus it follows that BVP (1.1) has a positive solution x^{**}. The proof is complete. □

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