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A NON-TRIVIAL LIMIT BEHAVIOUR OF PROPERLY NORMALIZED DELAYED RANDOM SUMS IN THE POWER NORMALIZATION

KOKKADA VIDYALAXMI AND GOOTY DIVANJI*

Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d positive valued asymmetric stable random variables with a common distribution function F with index $\alpha, 0 < \alpha < 1$. The present work intends to study the non-trivial limit behaviour for properly normalized delayed random sums in the power normalization and study the number of boundary crossing random variables as an application.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) positive valued asymmetric stable random variables (r.v.s) with a common distribution function F with index $\alpha, 0 < \alpha < 1$. Set $S_n = \sum_{k=1}^{n} X_k, n \geq 1$ and $T_{a_n} = \sum_{k=n+1}^{n+a_n} X_k = S_{n+a_n} - S_n$, where $a_n, n \geq 1$, is the non-decreasing function of the positive integers of n such that $0 < a_n \leq n$ for all n and $\frac{a_n}{n} \sim b_n$, where b_n is non-increasing. The sequence $\{T_{a_n}, n \geq 1\}$ is called a (forward) delayed sum sequence (see Lai(1973)).

Let $\{L_n, n \geq 1\}$ be a sequence of non-decreasing positive integer r.v.s., with finite mean, independent of $\{X_n, n \geq 1\}$ such that $\frac{L_n}{n}$ is non-increasing and $\lim_{n\to\infty} \frac{L_n}{n} = 1$ a.s. Now parallel to the delayed sums T_{a_n}, Divanji(2017) introduced delayed random sums as $M_{L_n} = \sum_{k=n+1}^{n+L_n} X_k = S_{n+L_n} - S_n$, and studied the non-trivial limit behavior of properly normalized delayed random sums under some conditions on L_n.

We foresee that the delayed random sums theory will have a possible applications in finance by taking $X_i's$ as price changes or log returns, which are separated by random waiting times L_n between trades. Another possible application has been stated by Sreehari and Chen(2020) in control charts with censored samples where the sample size on each occasion will be a random number.

Laws of iterated logarithm (LIL) for heavy-tailed r.v.s are different from those

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* Corresponding Author.

r.v.s attracted to the normal law. Here we have used power normalization and the resulting limit theorem is called Chover's form of LIL (see Chover(1966)). For further developments in Chover's form of LIL, see Divanji(2004).

Lai(1973) studied the behavior of classical LIL for delayed sums $\{T_{a_n}, n \geq 1\}$, with finite variance, at different $a_n's$. For independent, but not identically distributed strictly positive stable r.v.s, Vasudeva and Divanji(1993) studied the non-trivial limit behavior for delayed sums and Sreehari and Chen(2020) extended to a more general class of stable r.v.s. Also they studied a non-trivial limit behavior for delayed random sums.

Associated with the laws of iterated logarithm or strong laws, the study of the r.v giving the number of boundary crossings has become very vital, since this study reflects the precision of the laws of iterated logarithm or strong laws. This boundary crossing problems have been attracted and studied by various authors such as Slivka and Savero(1970), Slivka(1969) and so on.

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. r.v.s with a common d.f. F with mean μ and variance σ^2. Let $S_n = \sum_{k=1}^{n} X_k, n \geq 1$. Let $\{d_n, n \geq 1\}$ be any sequence of real constants. We say that $\{d_n, n \geq 1\}$ belongs to the upper class, if $P(S_n > d_n i.o) = 0$ and to the lower class, if $P(S_n > d_n i.o) = 1$. Suppose that $\{d_n, n \geq 1\}$ belongs to the upper class. It would be interesting to find the boundary crossing probability $P(S_n \geq d_n, \text{for some } n \geq m > 0)$. Such boundary crossing probabilities have a statistical application in power one tests of one-sided hypotheses in confidence sequences for the unknown parameters of parametric families of distributions. See for example Lai and Lan(1976) and references therein.

In brief, if $b(n, \varepsilon) = (1 + \varepsilon)(2n \; log \; log \; n)^{1/2}$,for $n \geq 3$, then by classic LIL due to Hartman and Wintner(1941) asserts that with probability one, the inequality $|S_n - n\mu| \geq b(n, \varepsilon)$ will hold for finitely many n-values when $\varepsilon > 0$ and for infinitely many n-values when $\varepsilon < 0$. Let $\{Y_n(\varepsilon), n \geq 3\}$ be a sequence of indicator r.v.s defined by,

$$Y_n(\varepsilon) = \begin{cases} 1, & |S_n - n\mu| \geq \sigma b(n, \varepsilon) \\ 0, & \text{otherwise} \end{cases}$$

Let $\{N_m(\varepsilon), m \geq 3\}$ be the corresponding sequence of partial sums, i.e., $N_m(\varepsilon) = \sum_{n=3}^{m} Y_n(\varepsilon)$, for $m \geq 3$ and observe that $N_\infty(\varepsilon) = \sum_{n=3}^{\infty} Y_n(\varepsilon)$ which denotes the number of times the r.v $|S_n - n\mu|$ crosses or exceeds the boundary $b(n, \varepsilon)$. Also if $N_\infty(\varepsilon) = \sum_{n=3}^{\infty} Y_n(\varepsilon)$, then the LIL asserts that $P(N_\infty(\varepsilon) < \infty) = 1$ for $\varepsilon > 0$, while $P(N_\infty(\varepsilon) = \infty) = 0$ for $\varepsilon < 0$. Which says that if $\varepsilon > 0, N_\infty(\varepsilon)$ is a proper r.v or equivalently $N_\infty(\varepsilon)$ has proper distribution. Notice that $E(N_\infty^0) = P(N_\infty(\varepsilon) < \infty) = 1$. The 0^{th} moment of $N_\infty(\varepsilon)$ exists and hence the question arises whether $N_\infty(\varepsilon)$ possesses any moments of positive order. In the last section, we study the moments of these boundary crossing r.v.s related to LIL cosidered above.

The present work intends to obtain the non-trivial limit behavior to properly normalized delayed random sums in the power normalization and study the number of boundary crossings for the non-trivial limit behavior for delayed random sums.

Throughout this chapter, C, ε(small), k(integer) and n(integer), with or without a suffix or super suffix stand for positive constants, whereas a.s. and i.o. mean almost sure and infinitely often respectively and $g(x) \sim h(x)$ to stand for $\lim_{x\to\infty} \frac{g(x)}{h(x)} = 1$ a.s.

In the next section, we present the main result. Some boundary crossings problem for delayed random sums is obtained in the last section.

2. A Non-trivial limit behavior of delayed random sums

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. positive valued asymmetric stable r.v.s., with a common d.f. F, with index $\alpha, 0 < \alpha < 1$. Let $\{L_n, n \geq 1\}$ be a sequence of non-decreasing positive integer r.v.s. with finite mean, independent of $\{X_n, n \geq 1\}$ such that $\frac{L_n}{n}$ is non-increasing and $\lim_{n\to\infty} \frac{L_n}{n} = 1$ a.s. Let $\gamma_n = (\log \frac{n}{L_n} + \log\log n)$. Then $\liminf_{n\to\infty} \left\{ \frac{M_{L_n}}{L_n^{1/\alpha}} \right\}^{1/\gamma_n} = 1 a.s.$, where $M_{L_n} = S_{n+L_n} - S_n$.*

Proof. To prove the assertion, it is enough to show that for any sufficiently small $\varepsilon > 0$,

$$P\left(M_{L_n} \leq L_n^{1/\alpha} \left(\frac{n}{L_n} \ \ log\ n \right)^{-\varepsilon} i.o. \right) = 0 \tag{2.1}$$

and

$$P\left(M_{L_n} \leq L_n^{1/\alpha} \left(\frac{n}{L_n} \ \ log\ n \right)^{\varepsilon} i.o. \right) = 1. \tag{2.2}$$

From the condition $\lim_{n\to\infty} \frac{L_n}{n} = 1$ a.s., we have for any given $\varepsilon_1 > 0$, we can find some positive integer d such that

$$(1-\varepsilon_1)n \leq [(1-\varepsilon_1)n] \leq L_n \leq [(1+\varepsilon_1)n] \leq (1+\varepsilon_1)n,$$

for every $n \geq d$, where [x] denotes the largest integer contained in x. Which implies

$$u_n < L_n < v_n \text{ a.s.,} \tag{2.3}$$

where $u_n = (1-\varepsilon_1)n$ and $v_n = (1+\varepsilon_1)n$. Using (2.3), one can find some constants $C_1(> 0)$ and $C_2(> 0)$ such that,

$$L_n^{1/\alpha} \left(\frac{n}{L_n} \ \ log\ n \right)^{-\varepsilon} \leq C_1 n^{1/\alpha} (\log n)^{-\varepsilon} \tag{2.4}$$

and

$$L_n^{1/\alpha} \left(\frac{n}{L_n} \ \ log\ n \right)^{\varepsilon} \leq C_2 n^{1/\alpha} (\log n)^{\varepsilon} \tag{2.5}$$

where $C_1 = (1-\varepsilon_1)^{\varepsilon}(1+\varepsilon_1)^{1/\alpha}$ and $C_2 = (1-\varepsilon_1)^{-\varepsilon}(1+\varepsilon_1)^{1/\alpha}$. Using (2.4) and (2.5) in (2.1) and (2.2), we get,

$$P\left(M_{v_n} \leq C_1 n^{1/\alpha} (\log n)^{-\varepsilon} i.o. \right) = 0, \text{ where } M_{v_n} = S_{n+v_n} - S_n \tag{2.6}$$

and

$$P\left(M_{u_n} \le C_2 n^{1/\alpha}(\log n)^{\varepsilon} i.o.\right) = 1, \text{ where } M_{u_n} = S_{n+u_n} - S_n \tag{2.7}$$

The fact that X_n's are positive asymmetric stable r.v.s, implies that $\frac{M_{u_n}}{u_n^{1/\alpha}}$ and X_1 are identically distributed. Observe that $\frac{C_2 n^{1/\alpha}(\log n)^{\varepsilon}}{u_n^{1/\alpha}} = \frac{C_2 n^{1/\alpha}(\log n)^{\varepsilon}}{((1-\varepsilon_1)n)^{1/\alpha}} = C_3(\log n)^{\varepsilon} \to \infty$, as $n \to \infty$, where $C_3 = \frac{C_2}{(1-\varepsilon_1)^{1/\alpha}} > 0$ and hence we have,

$$\begin{aligned} P\left(M_{u_n} \le C_2 n^{1/\alpha}(\log n)^{\varepsilon}\right) &= P\left(X_1 \le C_3(\log n)^{\varepsilon}\right) \\ &\Leftrightarrow \lim_{n\to\infty} P\left(M_{u_n} \le C_2 n^{1/\alpha}(\log n)^{\varepsilon}\right) = 1 \end{aligned} \tag{2.8}$$

Notice that,

$$\begin{aligned} P\left(M_{u_n} \le C_2 n^{1/\alpha}(\log n)^{\varepsilon}\right) &= P\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} M_{u_m} \le C_2 m^{1/\alpha}(\log m)^{\varepsilon}\right) \\ &= \lim_{n\to\infty} P\left(\bigcup_{m=n}^{\infty} M_{u_m} \le C_2 m^{1/\alpha}(\log m)^{\varepsilon}\right) \\ &\ge \lim_{n\to\infty} P\left(M_{u_n} \le C_2 n^{1/\alpha}(\log n)^{\varepsilon}\right) \end{aligned}$$

From (2.8), we now get $P\left(M_{u_n} \le C_2 n^{1/\alpha}(\log n)^{\varepsilon} i.o.\right) = 1$. Hence proof of (2.7) is completed and proof for (2.2) follows from (2.7).

Now to prove (2.6), we define subsequence $n_k = \left[e^{k^b}\right], 0 < b < 1, k \ge 1$. Let $E_{1,n}, E_{2,k}$ and $E_{3,k}$ denote the events,

$E_{1,n} = \{M_{v_n} \le C_1 n^{1/\alpha}(\log n)^{-\varepsilon}$, $E_{2,k} = \left\{\inf_{n_k \le n \le n_{k+1}} M_{v_n} \le C_1 n^{1/\alpha}(\log n)^{-\varepsilon}\right\}$ and $E_{3,k} = \left\{S_{n_k+v_{n_k}} - S_{n_{k+1}} \le C_1 n_{k+1}^{1/\alpha}(\log n_{k+1})^{-\varepsilon}\right\}$.

Note that $\{E_{1,n}\ i.o\} \subset \{E_{2,k}\ i.o\} \subset \{E_{3,k}\ i.o\}$.

Hence, in order to prove(2.6) it is enough to prove that $P(E_{3,k} i.o) = 0$. (2.9)

We have, $P(E_{3,k}) = P\left(S_{n_k+v_{n_k}} - S_{n_{k+1}} \le C_1 n_{k+1}^{1/\alpha}(\log n_{k+1})^{-\varepsilon}\right)$. The fact that X_n's are i.i.d. positive asymmetric stable r.v.s implies that, $\frac{S_{n_k+v_{n_k}} - S_{n_{k+1}}}{(n_k+v_{n_k}-n_{k+1})^{1/\alpha}}$ and X_1 are identically distributed and hence, $P(E_{3,k}) = P(X_1 \le h_k)$, where $h_k = \frac{C_1 n_{k+1}^{1/\alpha}(\log n_{k+1})^{\varepsilon}}{(n_k+v_{n_k}-n_{k+1})^{1/\alpha}}$. Note that $v_{n_k} = (1+\varepsilon_1)n_k$ and $n_k = \left[e^{k^b}\right]$ and using this fact, we have $\frac{n_{k+1}^{1/\alpha}}{(n_k+v_{n_k}-n_{k+1})^{1/\alpha}} = \frac{n_{k+1}^{1/\alpha}}{(n_k+(1+\varepsilon_1)n_k-n_{k+1})^{1/\alpha}} = \frac{1}{\left(\frac{(2+\varepsilon_1)n_k}{n_{k+1}}-1\right)^{1/\alpha}}$

Consider $\frac{n_k}{n_{k+1}} = \frac{e^{k^b}}{e^{(k+1)^b}} \to 1$, as $k \to \infty$. Therefore $\frac{n_{k+1}^{1/\alpha}}{(n_k+v_{n_k}-n_{k+1})^{1/\alpha}} \to \frac{1}{((2+\varepsilon_1)-1)^{1/\alpha}} = \frac{1}{(1+\varepsilon_1)^{1/\alpha}} < 1 = (1-\varepsilon_2)$ (say), where $b < 1, \alpha < 1$ and $\varepsilon < 1$ which yields that $h_k \le C_4(1-\varepsilon_2)(\log n_k)^{-\varepsilon}$, where $C_4 > 0$.

Now using Theorem 1 of Feller(1966), page 448, there exists a constant $C_5 > 0$ such that

$$\begin{aligned} P(E_{3,k}) = P(X_1 \le h_k) &\le C_5 \exp\{-(1-\varepsilon_2)(\log n_k)^{-\varepsilon})^{-\alpha}\} \\ &\le C_5 \exp\{-(1-\varepsilon_2)^{-\alpha}(\log n_k)^{\alpha\varepsilon})\}. \end{aligned}$$

Since $(1-\varepsilon_2)^{-\alpha} > 1 = (1+\varepsilon_3)$ (say), for some $\varepsilon_3 > 0$,
$P(E_{3,k}) \leq C_5 \exp\{-(1+\varepsilon_3)(\log n_k)^{\alpha\varepsilon})\}$.
We now claim that, for some $\varepsilon_4 > 0, \exp\{(1+\varepsilon_3)(\log n_k)^{\alpha\varepsilon})\} = o\left(\frac{1}{(\log n_k)^{(1+\varepsilon_4)}}\right)$,
which implies, $\frac{\exp\{-(1+\varepsilon_3)(\log n_k)^{\alpha\varepsilon})\}}{\frac{1}{(\log n_k)^{(1+\varepsilon_4)}}} = \frac{(\log n_k)^{(1+\varepsilon_4)}}{\exp\{(1+\varepsilon_3)(\log n_k)^{\alpha\varepsilon})\}}$.
From the elementary knowledge on limits, we know that $\frac{\log n}{e^n} \to 0$ as $n \to \infty$ and from the fact that $(\log n_k)^{(1+\varepsilon_4)} \to \infty$ as $k \to \infty$ immediately implies that

$$\frac{(\log n_k)^{(1+\varepsilon_4)}}{\exp\{(1+\varepsilon_3)(\log n_k)^{\alpha\varepsilon})\}} \to 0 \text{ as } k \to \infty$$

and the claim is justified. Hence there exists $C_6 (> C_5)$ such that

$$P(E_{3,k}) \leq \frac{C_5}{(\log n_k)^{(1+\varepsilon_4)}} = \frac{C_5}{k^{b(1+\varepsilon_4)}}, \text{ since } n_k = \left[e^{k^b}\right] \text{ and } b < 1. \qquad (2.10)$$

Consequently, the series, $\sum_{k\geq 1} P(E_{3,k}) < \infty$, for sufficiently small $\varepsilon_4 (> 0)$, which in turn establishes (2.9) by appealing to Borel - Cantelli lemma and hence proof of (2.1) and (2.6) follows from (2.9). Thus, the proof of the Theorem is completed. □

3. Study of the number of boundary crossings related to the above LIL

Here we study the existence of moments of the number of boundary crossing r.v.s related to the above non-trivial limit behavior for properly normalized delayed random sums in the power normalization.

A similar study of the moments of $N_\infty(\varepsilon)$ is made for LIL results in the process. We express theorems of Slivka(1969) and Slivka and Savero(1970) in a more generalized form as a lemma, so that we can appeal to this lemma as and when needed.

Lemma 3.1. *Let $\{\xi_n, n \geq 1\}$ be a sequence of r.v.s and $\{A_n, n \geq 1\}$ be a sequence of sets on the real line $\Re$ such that $P(\xi_n \in A_n i.o) = 0$. Let $\{I(A_n), n \geq 1\}$ be a sequence of indicator r.v.s which are defined by,*

$$I(A_n) = \begin{cases} 1, & \text{if } \xi_n \in A_n \\ 0, & \text{otherwise} \end{cases}$$

and let $\{N_m, m \geq 1\}$ be the corresponding sequence of partial sums, i.e., $N_m = \sum_{n=1}^{m} I(A_n)$. From $N_\infty = \sum_{n=3}^{\infty} I_n(A_n)$, we know that $P(N_\infty < \infty) = 1$ or N_∞ is a proper r.v. Then for any $\lambda > 0$, $EN_\infty^\lambda < \infty$ whenever $\sum_{n=1}^{\infty} n^{\lambda-1} P(\xi_n \in A_n) < \infty$.

(The proof follows on similar lines of Slivka(1969) and Slivka and Savero(1970) and hence omitted).

Theorem 3.2. *$EN_\infty^\lambda < \infty$ whenever*
$\sum_{n=3}^{\infty} n^{\lambda-1} P\left(M_{L_n} \leq L_n^{1/\alpha}\left(\frac{n}{L_n}\log n\right)^{-\varepsilon}\right) < \infty$, *for all $0 < \lambda \leq 1$ and $\varepsilon > 0$.*

Proof. Define for any $\varepsilon > 0$,

$$Y_n(\varepsilon) = \begin{cases} 1, & M_{L_n} \le L_n^{1/\alpha} \left(\frac{n}{L_n}\log n\right)^{-\varepsilon} \\ 0, & \text{otherwise} \end{cases}$$

Let $N_m(\varepsilon)$ be the partial sum sequence of $Y_n(\varepsilon)$. i.e., $\sum\limits_{n=1}^{m} Y_n(\varepsilon)$, for $m \ge 3$, observe that equation (2.1) of the Theorem (2.1), $N_m(\varepsilon)$ is a proper r.v. Here we show that all the moments in $\lambda \in (0,1]$ are finite for this proper r.v. for which we establish the theorem for $\lambda = 1, EN_\infty < \infty$ and then claim that the existence of lower moments follows from higher moments . From the above Lemma 3.1 , identifying ξ_n with M_{L_n}, A_n with $\left(-\infty, L_n^{1/\alpha}\left(\frac{n}{L_n}\log n\right)^{-\varepsilon}\right]$, and N_∞ with $N_\infty(\varepsilon)$. By statement (2.1) of Theorem 2.1, we have $EN_\infty(\varepsilon) < \infty$ whenever $\sum\limits_{n=3}^{\infty} P\left(M_{L_n} \le L_n^{1/\alpha}\left(\frac{n}{L_n}\log n\right)^{-\varepsilon}\right) < \infty$. From (2.10) of Theorem 2.1, we can find some constant $C_6 > 0$ such that
$\sum\limits_{n=3}^{\infty} P\left(M_{L_n} \le L_n^{1/\alpha}\left(\frac{n}{L_n}\log n\right)^{-\varepsilon}\right) < C_6 \sum\limits_{k=k_4}^{\infty} \frac{1}{k^{b(1+\varepsilon_4)}} < \infty$, where $b < 1$ and $n_k = \left[e^{k^b}\right]$, which proves that $EN_\infty(\varepsilon) < \infty$, for $\lambda = 1$ and therefore $EN_\infty^\lambda < \infty$, for all $0 < \lambda \le 1$. Hence the proof the Theorem is completed. $\square$

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References

1. Chover,J: A law of iterated logarithm for stable summands, *Proc. Amer.Math.Soc* **17** (1966) 441–443.
2. Feller,W: An introduction to probability theory and its applications, *Wiley Eastern Limited , Fourth Wiley Eastern reprint* **11** (1966) .
3. Gooty Divanji: A law of iterated logarithm for delayed random sums, *Research and Reviews:J. Statist* **6** (2017) 24–32.
4. Gooty Divanji: A law of iterated logarithm for subsequences of partial sums which are in the domain of partial attraction of a semi stable law , *Probability and Mathematical Statistics, Fasc.2,41* **24** (2004) 433–442.
5. Hartman,P and Wintner,A: On the law of iterated logarithm, *American Journal of Mathematics* **63(1)** (1941) 169–176.
6. Lai,T.L.: Limit theorems for delayed sums, *Ann. Probab.* **2** (1973) 432–440.
7. Lai,T.L. and Lan,K.K.:On the last time and boundary crossings related to the strong law of large numbers and the law of iterated logarithm, *Zeitschrift fur Wahrscheinlichkeitstheorie und Verwandte Gebiete* **34(1)** (1976) 59–71.
8. Slivka,J: On the law of iterated logarithm, *Proc. National Academy of Sciences* **63(2)** (1969) 289–291.
9. Slivka,J and Savero,N.C: On the strong law of large numbers, *Proc.American Mathematical Society* **24(4)** (1970) 729–734.
10. M.Sreehari and P.Chen: Study of the limiting behavior of delayed random sums under non-identical distributions setup and a Chover type LIL, *Theor.Probability and Math. Statist.* **100** (2020) 150–168.
11. Vasudeva,R. and Divanji,G.: The law of iterated logarithm for delayed sums under a non-identically distribution setup, *Theory Probab. Appl.* **37(3)** (1993) 497–506.

Kokkada Vidyalaxmi: Department of Studies in Statistics, Manasagangotri, University of Mysore, Mysuru - 570 006, Karnataka - India
Email address: vidyalaxmi.k@gmail.com

Gooty Divanji: Department of Studies in Statistics, Manasagangotri, University of Mysore, Mysuru - 570 006, Karnataka - India
Email address: gootydivan@yahoo.co.in